Revenue-Maximizing Mechanism in Bilateral Trade

Zhonghong Kuang†  Weiran Shen†  Fan Wu§

June 9, 2023

Abstract

We study the revenue-maximizing mechanism in bilateral trade with interdependent valuation. The seller privately observes the item’s quality, while the buyer privately knows his type of valuation. The seller’s valuation only depends on the quality, and the buyer’s valuation depends on both the quality and his type. The trade goes through an uninformed mediator. The mediator ex-ante designs a mechanism to collect or disclose information, as well as sell information and levy fees. Information exchange or sales may occur over multiple rounds, and players can exit anytime. We characterize a simple direct mechanism that maximizes revenue. In this mechanism, both players report their types, and the mediator directly recommends whether to trade. Moreover, the recommendation rule takes a threshold structure.

Keywords: Mechanism Design; Revenue Maximizing; Bilateral Trade

JEL Codes: C73, D82, D83

*For helpful comments, we are grateful to Luciano Pomatto, Fedor Sandomirskiy, Omer Tamuz. We thank Zhikang Fan for excellent research assistance.

†Renmin University of China; email: kuang@ruc.edu.cn.

‡Renmin University of China; email: shenweiran@ruc.edu.cn.

§Division of the Humanities and Social Sciences, California Institute of Technology; email: fwu2@caltech.edu.
1 Introduction

Bilateral trade plays a crucial role in mechanism design, often involving the presence of a mediator or intermediary. In the realm of finance, large mergers and acquisitions (M&A) are often facilitated by investment banks, who act as intermediaries and charge a fee ranging from 1% to 10%. A notable example is the acquisition of Swedish Volvo by Geely, a Chinese automobile company, from Ford Motor in 2010, where a financial intermediary played a central role. The scale of M&A activities continues to expand, emphasizing the growing importance of financial intermediaries. According to Refinitiv, global M&A volume reached a historical peak of 5.8 trillion dollars in 2021.¹

Another scenario where intermediaries are prevalent is the sale of lab’s patents by technology institutes to innovative companies. For instance, Tsinghua University mandates that all laboratory research outputs be sold to enterprises through the Office of Technology Transfer. In this arrangement, the laboratory retains about 70% of the revenue, while the remaining 30% goes to Tsinghua. Similar fee structures exist in other universities and research institutions. In 2020, the total value of patents sold through Chinese universities was approximately 18 billion dollars.²

A common feature of bilateral trade is the interdependence of valuations between the buyer and the seller. The valuation of an item is influenced by its quality, with higher quality leading to higher valuations. However, typically only the seller possesses information about the item’s quality. For instance, Geely had uncertainty about Volvo’s manufacturing technology before the M&A took place. In the largest M&A deal ever in the United States, when American Online acquired Time Warner, American Online had an informational disadvantage regarding the TV and film business compared to the owners of Time Warner. Similarly, when an enterprise purchases a patent from an institute, it may have limited knowledge about the quality of the technology held by the laboratory.

Additionally, the interaction between the buyer, mediator, and seller is dynamic. Either party can choose not to proceed with the trade at any point before the transaction is finalized. Meanwhile, the mediator possesses considerable flexibility in designing the trading

mechanism, including exchanging or selling information between the two parties. In the context of M&A, investment banks are well-positioned to gather relevant information from both parties. Notably, the mediator, usually the investment bank, takes the responsibility of conducting due diligence, which is often not carried out by the buyer. For another example, many online marketplaces charge customers fees to upgrade to VIP, granting them access to more information.

Given these circumstances, it is crucial to determine how the mediator should design a dynamic trading mechanism to maximize revenue, when valuations are interdependent. Yet, in the mechanism design literature, problems with interdependent valuations are notoriously hard to solve. We construct a tractable model suited for dynamic problems with interdependent valuations.

Formally, we examine a bilateral trade scenario involving two players: a seller and a buyer. The seller possesses an item for sale, which the buyer intends to purchase. The buyer’s type of valuation \( t \) is his private information, while the seller privately knows the item’s quality \( q \). The seller’s valuation of the item \( r(q) \) only depends on the quality. The buyer’s valuation \( v(t, q) \) depends on both his type and the quality. Here, the type can be interpreted as the market-related information in the M&A example and patent selling example. The buyer is better informed about the market condition when buying a patent or acquiring a company.

The trade is conducted through an uninformed mediator. The mediator ex-ante designs a mechanism that specifies how the transaction will be conducted. The mediator can communicate with each player in the mechanism to collect or disclose information, as well as sell information and levy fees. Information exchange or sales may occur over multiple rounds. In the end, the mediator recommends the trade with posted prices, or alternatively, no trade, which would end the transaction. The mediator ex-ante commits to his strategy in the mechanism. Both players are free to exit the mechanism at any stage.

Due to the mediator’s extensive flexibility in the mechanism design, solving for a revenue-maximizing mechanism seems daunting. To address this, we define a class of simple direct mechanisms in which players report their types, and the mediator recommends whether to trade or not. Our first result (Proposition 1) shows that for any general mechanism with any weak PBE induced by the mechanism, there exists a direct mechanism that generates the
same revenue. This means that, optimally, the mediator does not need to design complex mechanisms that involve information transmission, belief manipulation, or selling information. All that is required is a recommendation to trade, from which the mediator derives revenue.

We then characterize the revenue-maximizing direct mechanism. We propose a novel generalized virtual value function. We show that a direct mechanism maximizes revenue and takes a threshold structure: the mediator recommends trade if and only if the buyer’s generalized virtual value is above the seller’s virtual cost (Proposition 2). Moreover, we derive the players’ payment function in closed form. Our technical contribution is a novel ironing technique (Proposition 3). The standard ironing procedure introduced by Myerson (1981) fails in our model as players’ valuations are interdependent.

Related Literature

Our paper contributes to the literature on bilateral trade problems. Myerson and Satterthwaite (1983) characterize revenue-maximizing mechanisms in bilateral trade. Their game is static, i.e., both the seller and the buyer cannot exit the mechanism after participation. In our setting, players can exit the mechanism at any time, thus rendering the revelation principle based on Bayesian Nash equilibrium inapplicable. Recent studies have explored variants of the classical bilateral trade model. Eilat and Pauzner (2021) assume that the mediator cannot deny efficient trade, while Čopić and Ponsatí (2016) consider a robust design problem without a prior. In all three studies above, the buyer and seller’s valuations are independent. In contrast, we assume that the seller’s private information is the quality of the item, and that the buyer’s valuation depends on this quality. This natural interdependent valuation setting is more general and expands the applicability of the bilateral trade model.

Mao et al. (2022) consider a setting in which both a buyer and a seller can establish a communication protocol beforehand, and both have commitment power. In this protocol, the players send verifiable signals to each other in turn. However, in our setting, only the mediator has commitment power.

Our paper also contributes to the literature on information design (Kamenica and Gentzkow,
2011; Bergemann and Morris, 2016) with a mediator who has commitment power (Arieli et al., 2022; Kosenko, 2022; Kuang et al., 2019). Schottmüller (2023) studies information design in the standard bilateral trade model of Myerson and Satterthwaite (1983), assuming that neither the seller nor the buyer is informed about their valuation.

Our paper also relates to the literature on selling information in bilateral trade. Ali et al. (2022) study how an intermediary designs and sells hard information to robustly maximize its revenue. In their model, the intermediary can use a statistical test that partially reveals the state and can control the accuracy of the test. Bergemann et al. (2018) consider scenarios in which the seller also sells information about the item. In our model, the mediator is a third party with no information advantage, facing two privately informed players.

2 The Model

**Primitives.** Consider a bilateral trade setting with two players, a seller (she) $s$ and a buyer (he) $b$. The seller has an item for sale which the buyer seeks to buy. Let $t \in T$ denote the buyer’s type of valuation, which is his private information. Denote by $q \in Q$ the quality of the item, which is the seller’s private type. We assume $T$ and $Q$ are compact intervals in $\mathbb{R}$, i.e., $T = [t_1, t_2]$, $Q = [q_1, q_2]$.

Let $r(q)$ denote the seller’s reserve valuation for the item. We assume $r$ is continuous and strictly increasing in $q$. Without loss of generality, we normalize $r(q) = q$.\(^3\) Let $v(t, q)$ be the buyer’s valuation with type $t$ given item quality $q$. The higher the buyer’s type, the more he values the item. We hence assume that $v(t, q)$ is increasing and differentiable in $t$. Our setting is general and nests Myerson and Satterthwaite (1983) where $v(t, q)$ only depends on $t$, Akerlof (1970) where $v(t, q)$ only depends on $q$.

Both $q$ and $t$ are random variables independently drawn from publicly known distributions $G(q)$ and $F(t)$ with full support on $Q$ and $T$. Suppose that both $G(q)$ and $F(t)$ are differentiable, and $g(q)$ and $f(t)$ are the corresponding probability density functions.

**Mechanisms.** The trade must be transacted through an uninformed mediator $me$. The mediator ex-ante designs a mechanism that specifies how the transaction will be conducted.

\(^3\)Given quality $\tilde{q}$ and $r(\tilde{q})$, we can redefine the quality $q \triangleq r(\tilde{q})$. 

The mediator can communicate with each player in the mechanism to collect or disclose information, as well as sell information and levy fees. Information exchange or sales may occur over multiple rounds. In the end, the mediator recommends the trade with posted prices, or alternatively, no trade, which would end the transaction. The mediator ex-ante commits to his strategy in the mechanism.\(^4\) Both players are free to exit the mechanism at any stage. Formally,

**Definition 1.** A *mechanism* consists of finite, say \(n\), communication rounds and one trade round. Each communication round is *private*, i.e., the interaction between the mediator and one player is unobservable to the other player. Let \(I_{me}^k \in \mathcal{I}_{me}^k\) denote the mediator’s information set at the end of \(k\)th communication round,\(^5\) where \(\mathcal{I}_{me}^k\) is the set of information sets. In the \(k\)th round of communication, the following three stages occur sequentially.

1. Each player \(i \in \{s, b\}\) sends a messages \(m_i^k \in M_i^k\) to the mediator. We allow for \(M_i^k\) to be a singleton, which shuts down player \(i\)’s communication at this stage.

2. For each player \(i \in \{s, b\}\), the mediator sends a message \(m_{me,i}^k \in M_{me,i}^k\) and offers a menu \(x_i^k \in X_i^k\). A menu is a set. Each element in this set includes a description of the information offered in the next stage and its fee. All information is a garbling of the mediator’s current information set.\(^6\) The mediator’s strategy is \(S_{me,i}^k : \mathcal{T}_{me}^{k-1} \times M_b^k \times M_s^k \mapsto \Delta(M_{me,i}^k \times X_i^k)\). The message \(m_{me,i}^k\) is free, while the information in the menu is costly. We allow for \(M_{me,i}^k\) to be a singleton to shut down this communication. We also allow for all menus to only contain no information with no fee to shut down information trading.

3. Each player \(i \in \{s, b\}\) chooses an item \(\nu_i^k \in x_i^k\) and pay the fee. Then, information realization \(r_i^k\) is generated accordingly.

In the trade round,

\(^4\)Our mechanism generalizes the “generic interactive protocol” proposed by Babaioff et al. (2012).
\(^5\)We have \(I_{me}^0 = \emptyset\).
\(^6\)Mathematically, all information is an output of a Blackwell experiment with the mediator’s information set as input.
1. The mediator announces trade or no trade. If “trade”, the mediator posts prices $P_b \in \mathbb{R}$ and $P_s \in \mathbb{R}$ to the buyer and the seller. The price pair $(P_b, P_s)$ is generated according to the rule $T_{me}^n \mapsto \Delta(\mathbb{R}^2)$. If “no trade”, the mechanism terminates.

2. The buyer and the seller decide whether to accept the trade. If both accept, the trade occurs; the buyer pays $P_b$ to the mediator, and the mediator pays $P_s$ to the seller. If anyone rejects, the mechanism terminates.

At any stage, if any player $i \in \{s, b\}$’s action is not permissible by the mechanism, like no response, the mechanism terminates. Thus players are free to exit at any time.

Preferences. We assume a quasi-linear preference for both players. We let $z$ denote the entire history of players’ actions. Denote by $p_b(z)$ the total payment of the buyer along the history $z$, $p_s(z)$ the total payment the seller receives along the history $z$. The buyer’s payoff is

$$u_b(z) = v(t, q) \cdot 1_{\text{Trade}} - p_b(z).$$

The seller’s payoff is

$$u_s(z) = -r(q) \cdot 1_{\text{Trade}} + p_s(z).$$

Solution Concept. Our solution concept is Perfect Bayesian Equilibrium (PBE). To formally define it, we need to define the extensive-form game (between the buyer and the seller) induced by the mechanism. We relegate the definition of the game to Appendix A. Here we briefly define PBE.

Let $H_i$ denote the sets of histories where player $i \in \{s, b\}$ moves. $\mathcal{T}_i$ is a partition of $H_i$ such that an element $I \in \mathcal{T}_i$ is called an information set of player $i$. A strategy $S_i(I)$ of player $i \in \{s, b\}$ is a function that assigns a probability distribution over actions $A(I)$ to each information set $I \in \mathcal{T}_i$. A belief is a function $\mu_i : \mathcal{T}_i \mapsto [0, 1]$ that assigns to each information set $I \in \mathcal{T}_i$ a probability such that the probabilities of the node in any information set sum up to 1, i.e., $\sum_{h \in I} \mu_i(h) = 1, \forall I \in \mathcal{T}_i$, for both players $i \in \{s, b\}$.

An assessment is a pair $(S, \mu)$, where $S$ is a strategy profile and $\mu$ is a belief profile. Let $Z_I$ be the set of all terminal histories $z$ reachable from some nodes in $I$, and $Prob(z|S, \mu, I)$

---

7 The mediator’s information set $T_{me}^k$ is constructed by appending $T_{me}^{k-1}$ with $(m_{i}^{k}, m_{me,i}^{k}, s_{i}^{k}, \nu_{i}^{k}, r_{i}^{k})$ for all $i \in \{s, b\}$. 

7
be the conditional probability of reaching terminal history \( z \in Z_i \) given an assessment \((S, \mu)\) and information set \( I \in \mathcal{I}_i \). Then the conditional expected payoff of player \( i \) given \((S, \mu)\) at information set \( I \) is
\[
U_{i,I}(S|\mu) = \sum_{z \in Z_i} \text{Prob}(z|S, \mu, I)u_i(z).
\]

**Definition 2.** An assessment \((S^*, \mu^*)\) is a weak perfect Bayesian equilibrium if the following conditions hold for each player \( i \in \{s, b\} \).

- *Bayes’ rule.* \( \mu_i^* \) satisfies Bayes’ rule whenever possible.
- *Sequential rationality.* For every information set \( I \in \mathcal{I}_i \) and every strategy \( S_i \),
\[
U_{i,I}(S_i^*, S_{-i}^* | \mu^*) \geq U_{i,I}(S_i, S_{-i}^* | \mu^*);
\]

### 3 Direct Mechanisms

The problem of deriving the revenue-maximizing mechanism seems daunting, as our mechanism space is quite large. There is no bound on how many rounds the mediator can communicate with both players, no restrictions on what information the mediator can collect or reveal for free, and no constraints on the menu design, i.e., on what type of information the mediator can sell and how much to charge.

Yet, we show that, from a revenue-maximizing perspective, it is without loss of generality to focus on direct mechanisms, where players truthfully report types and then the mediator recommends trade or not. Here, the mediator neither exchanges nor sells information, and he only charges players when the trade occurs.

**Definition 3** (Direct Mechanism). A *direct* mechanism consists of a message space \( M = \{\text{trade, no trade}\} \), the probability of sending the message ”trade” \( \pi : T \times Q \mapsto [0, 1] \), the buyer’s payment function \( P_b : T \mapsto \mathbb{R} \), the payment to the seller \( P_s : Q \mapsto \mathbb{R} \) and proceeds as follows:

1. The mediator announces and commits to \( \pi, P_b \) and \( P_s \).
2. The buyer and seller report their types \( t \in T \) and \( q \in Q \) to the mediator privately.
3. A message realizes according to $(\pi(t, q), 1 - \pi(t, q))$.

4. The players decide whether to obey if the message is “trade”. Otherwise, the mechanism terminates.

5. If both players accept, the trade occurs, the buyer pays $P_b(t)$, and the seller receives $P_s(q)$. If anyone rejects, the mechanism terminates.

We say that a direct mechanism is feasible if both players are willing to participate (IR), report their types truthfully (IC), and obey the mediator’s recommendation (Obedience). Notice that in a direct mechanism, obedience is equivalent to the IR constraint, as the only pivotal event in participating in a direct mechanism is when the mediator recommends trade. We now show that it is without loss of generality to focus on direct mechanisms.

**Proposition 1** (Revenue Equivalence). For any general mechanism and a corresponding weak PBE $(S^*, \mu^*)$, a direct mechanism with

$$
\pi(t, q) \triangleq \text{Prob}(\text{trade occurs}|S^*, t, q),
$$

$$
P_b(t) \triangleq \frac{E_{q \sim g}[E_z(p_b(z)|S^*, t, q)]}{E_{q \sim g}[\pi(t, q)]},
$$

$$
P_s(q) \triangleq \frac{E_{t \sim f}[E_z(p_s(z)|S^*, t, q)]}{E_{t \sim f}[\pi(t, q)]},
$$

is feasible and yields the same expected revenue and the same expected payoff for both players.

**Proof.** It suffices to prove the equivalence for the buyer, as the case for the seller and mediator is similar. The conclusion is trivial when $\int_q \pi(t, q)g(q) = 0$. Now suppose $\int_q \pi(t, q)g(q) > 0$.

We first characterize the buyer’s expected payoff in the direct mechanism, when both players truthfully report and obey the mediator. Let $g(q|t, m = \text{trade})$ be the buyer’s belief

\[\int_{q \in Q} g(q|m = \text{trade}, t)v(t, q) \, dq - P_b(t) \geq 0.\]

Multiplying $\int_{q \in Q} \pi(t, q)g(q) \, dq \geq 0$ on both sides, we obtain the buyer’s IR

\[\int_{q \in Q} \pi(t, q)[v(t, q) - P_b(t)]g(q) \, dq \geq 0.\]
of the quality upon receiving message \( m = \text{trade} \). We have

\[
g(q|t, m = \text{trade}) = \frac{\pi(t, q)g(q)}{\int_q \pi(t, q)g(q) \, dq}.
\]

If \( m = \text{trade} \), the buyer’s expected payoff is

\[
E_{q \sim g(q|t, m = \text{trade})} [v(q, t)] - P_b(t) = \frac{\int_q \pi(t, q)g(q)v(q, t) \, dq - E_{q \sim g} [E_z(p_b(z)|S^*, t, q)]}{\int_q \pi(t, q)g(q) \, dq}.
\]

When receiving message \( m = \text{no trade} \), the buyer’s payoff is 0. Therefore, the expected payoff of the buyer with type \( t \) is

\[
\int_q \pi(t, q)g(q) \, dq [E_{q \sim g(q|t, m = \text{trade})} v(q, t) - P_b(t)]
\]

\[
= \int_q \pi(t, q)g(q)v(q, t) \, dq - E_{q \sim g} [E_z(p_b(z)|S^*, t, q)]
\]

\[
= E_{q \sim g} [\pi(t, q)v(q, t) - E_z(p_b(z)|S^*, t, q)]
\]

\[
= E_{q \sim g} [\text{Prob(trade occurs}|S^*, t, q)v(q, t) - E_z(p_b(z)|S^*, t, q)]
\]

where the last term is the buyer’s expected payoff in the general mechanism.

Now we show this direct mechanism is feasible. Players shall truthfully report in the direct mechanism, as \((S^*, \mu^*)\) forms a PBE in the general mechanism, i.e., any player of one type has no strict incentive to mimic another type’s strategy in the general mechanism. The IR constraint holds in the direct mechanism since players are willing to participate in the original general mechanism.

We now illustrate the intuition. In any game induced by a general mechanism, an equilibrium must satisfy both the individual rationality (IR) and a generalized incentive compatibility (IC) constraint. Specifically, players must obtain non-negative payoffs from their participation, and no player can profitably mimic the strategy of another type. Therefore, the IC and IR constraints restrict the mediator’s achievable revenue across all mechanisms. Granting the mediator extra flexibility to communicate or sell information cannot help the
mediator overcome these constraints.

Our result resembles but differs from the revelation principle. The standard revelation principle in Myerson (1982) does not apply as we consider dynamic games. Moreover, our result differs from the communication revelation principle (Forges, 1986; Sugaya and Wolitzky, 2021).  

**Remark 1.** The Revenue Equivalence Proposition is strong as it holds for any weak PBE $\left(S^*, \mu^*\right)$ in any general mechanism. That is, we do not put any constraint on the belief off the equilibrium path in $\left(S^*, \mu^*\right)$. In contrast, the PBE in the direct mechanism is always on the equilibrium path, as we assume that $F(t)$ and $G(q)$ have full support.

Although the mediator is free to choose any general mechanism, our Revenue Equivalence Proposition simplifies the revenue-maximizing mechanism design problem. Optimally, the mediator does not need to design intricate mechanisms to engage in information transmission, belief manipulation, or selling information. All he needs to do is to recommend trade from which he derives revenue. For the rest of the paper, we focus on direct and feasible mechanisms.

## 4 Revenue-Maximizing Mechanism

In this section, we characterize the revenue-maximizing direct mechanism. The mediator designs $\pi(t, q)$, $P_b(t)$ and $P_s(q)$ to maximize revenue

$$\int_{q \in Q} \int_{t \in T} \pi(t, q)[P_b(t) - P_s(q)] f(t) g(q) \, dt \, dq.$$  

The communication RP states that: in any game (induced by a mechanism), any distribution of outcomes in any equilibrium also arises in a canonical equilibrium where players communicate their private information. In this result, the “direct” and general mechanism only differ in their message space.

The Revenue Equivalence Proposition 1 differs from communication RP in two aspects. First, Proposition 1 does not require the distribution of outcomes in the general and the direct mechanism to coincide. Although the trading probability conditional on a type profile $(t, q)$ coincides, the distribution of payment might differ. In a general mechanism, the mediator could obtain revenue by selling information (which leads to no trade), while the payment in a direct mechanism can only come from trading. Second, in communication RP, the “direct” and general mechanism differ only in their message space. But in our model, the direct mechanism simplifies the general mechanism so drastically that changes the players’ action set. In the general mechanism, players can exchange and buy information for many rounds, while players only report their types once in the direct mechanism.
under the feasibility constraint.

**Individual Rationality (IR).** This constraint ensures that the players participate in the mechanism voluntarily. Both players’ payoff is 0 if they do not participate. The buyer’s expected payoff when participating in the direct mechanism must be above 0.

\[
U_b(t) = \mathbb{E}_{q \sim G}[\pi(t, q)(v(t, q) - P_b(t))]
= \int_{q \in Q} \pi(t, q)[v(t, q) - P_b(t)]g(q) \, dq \geq 0. \tag{1}
\]

Similarly, we have the IR constraint for the seller.

\[
U_s(q) = \mathbb{E}_{t \sim F}[\pi(t, q)(P_s(q) - r(q))]
= \int_{t \in T} \pi(t, q)[P_s(q) - r(q)]f(t) \, dt \geq 0. \tag{2}
\]

**Incentive Compatibility (IC).** Both players must truthfully report their types. The type-\(t\) buyer’s expected payoff when misreporting \(t'\) is

\[
U_b(t'; t) \triangleq \int_{q \in Q} \pi(t', q)[v(t, q) - P_b(t')]g(q) \, dq.
\]

We also allow for double deviations, i.e., after misreporting, the buyer can disobey the trade recommendation. So the buyer’s expected deviation payoff is \(\max\{U_b(t'; t), 0\}\). Thus the buyer’s IC constraint is

\[
U_b(t) \geq \max\{U_b(t'; t), 0\}.
\]

Given the IR constraint, we can drop the zero term and write IC as

\[
U_b(t) \geq U_b(t'; t). \tag{3}
\]

Similarly, for the seller, we have

\[
U_s(q'; q) \triangleq \int_{t \in T} \pi(t, q')[P_s(q') - r(q)]f(t) \, dt
\]
and the IC

\[ U_s(q) \geq U_s(q'; q). \]  

(4)

We formulate the feasible revenue-maximizing mechanism design problem as follows:

\[
\begin{align*}
\text{maximize} & \quad \int_{q \in Q} \int_{t \in T} f(t)g(q)\pi(t, q)[P_b(t) - P_s(q)] \, dt \, dq \\
\text{subject to} & \quad U_b(t) \geq 0, & \forall t \in T \\
& \quad U_s(q) \geq 0, & \forall q \in Q \\
& \quad U_b(t) \geq U_b(t'; t), & \forall t, t' \in T \\
& \quad U_s(q) \geq U_s(q'; q), & \forall q, q' \in Q
\end{align*}
\]

(5)

We now solve for this problem in closed form. Let \( \phi_s(q) \) be the seller’s virtual cost.\(^{10}\)

Moreover, we define the buyer’s \textit{generalized virtual value} \( \phi_b(t, q) \).

\[
\phi_b(t, q) \triangleq v(t, q) - \frac{1 - F(t)}{f(t)} \frac{\partial v(t, q)}{\partial t}, \quad \phi_s(q) \triangleq q + \frac{G(q)}{g(q)}.
\]

As any quality level \( q \), we define the \textit{break-even type}\(^{11}\)

\[ t(q) \triangleq \sup\{ t \in T | \phi_b(t, q) \leq \phi_s(q) \}. \]

It is the type of the buyer whose virtual valuation equals the seller’s virtual cost.

We first deal with the case when both the generalized virtual value and the break-even type are increasing in type \( t \).

\textbf{Definition 4} (Regularity). A problem is \textit{regular} if the break-even type \( t(\cdot) \) and the buyer’s virtual value \( \phi_b(\cdot, q) \) for all \( q \) are non-decreasing.

The regularity condition is satisfied in many cases. First, when the buyer’s valuation only depends on the buyer’s type \( v(t, q) = t \), regularity boils down to (Myerson and Satterthwaite,\(^{12}\))

---

\(^{10}\)The definition of the virtual cost function is standard in the literature (see, e.g., Manelli and Vincent (1995); Myerson and Satterthwaite (1983)).

\(^{11}\)This type might not exist and we adopt the convention that \( t(q) = t_1 \) if \( \phi_b(t, q) > \phi_s(q) \) for all \( t \in T \).
increasing virtual type \( \psi(t) \triangleq t - \frac{1 - F(t)}{f(t)} \) and virtual cost \( \phi_s(q) \). Well-known conditions like non-decreasing hazard rate \( \frac{f(t)}{1 - F(t)} \) are sufficient for \( \psi(t) \) to be increasing. Second, when \( v(t, q) \) only depends on the quality \( q \), regularity holds when \( \phi_s(q) \) does not cross \( v(q) \) or single crosses \( v(q) \) from below. Third, when the buyer’s valuation is additively separable \( v(t, q) = t + \alpha(q) \), regularity holds if virtual type \( \psi(t) \) and \( \phi_s(q) - \alpha(q) \) are increasing. We handle the irregular case in the next section.

We define the weighted trading probability

\[
\Pi_b(t; \pi) \triangleq \int_{q \in Q} \frac{\partial v(t, q)}{\partial t} \pi(t, q) g(q) \, dq, \quad \Pi_s(q; \pi) \triangleq \int_{t \in T} \pi(t, q) f(t) \, dt.
\]

Note that \( \Pi_b(t; \pi) \) and \( \Pi_s(q; \pi) \) can be viewed as the expected probability of trade, except that \( \Pi_b(t; \pi) \) is weighted by \( \frac{\partial v(t, q)}{\partial t} \). Now we are ready to present our revenue-maximizing mechanism. The mechanism has a simple threshold structure.

**Proposition 2.** Suppose the regularity condition holds. Then the following direct mechanism is feasible and maximizes revenue.

\[
\pi^*(t, q) = \begin{cases} 
1 & \text{if } \phi_b(t, q) \geq \phi_s(q) \\
0 & \text{otherwise}.
\end{cases}
\]

\[
P_b^*(t) = \mathbb{E}_q(v(t, q)|\pi^*(t, q) = 1) - \frac{1}{\mathbb{E}_q(\pi^*(t, q))} \int_{t_1}^t \Pi_b(x; \pi^*) \, dx, \quad (6)
\]

\[
P_s^*(q) = \mathbb{E}_t(r(q)|\pi^*(t, q) = 1) + \frac{1}{\mathbb{E}_t(\pi^*(t, q))} \int_{q_2}^q \Pi_s(x; \pi^*) \, dx. \quad (7)
\]

In the payment function for both players, the first term represents the expected valuation conditional on trade, and the second term represents the player’s expected payoff conditional on trade. The integral in the second term represents the players’ ex-ante payoff from participating in the mechanism, which is also the smallest payoff the mechanism must offer to the players under the IC and IR constraint. We can see that the revenue-maximizing mechanism must yield \( U_b(t_1) = U_s(q_2) = 0 \), i.e., zero payoff for the lowest type buyer and the highest

\[\text{Suppose a mechanism } (\pi, P_b, P_s) \text{ is feasible and yields } U_b(t_1) > 0. \text{ We can construct a new mechanism} \]
type seller.

Now suppose the valuation gap \( v(q, t) - r(q) \) increases in the quality \( q \). As the quality rises, the gains from trade increases. We expect the mediator to gain more by facilitating the transaction more often. However, the trade occurs if and only if \( t \geq t(q) \), where \( t(\cdot) \) is increasing. This implies that the higher the quality, the lower the trading probability. Why does the mediator shut down the trade more often when the gains from trade are larger?

The answer lies in the IC constraint. Suppose the seller’s weighted trading probability \( \Pi_s(q) \) is increasing. Consider two possible qualities \( \overline{q} > q \). For the seller with quality \( \overline{q} \) to report truthfully, the expected payment difference between quality \( \overline{q} \) and \( q \) must be enough to compensate for losing the item \( \overline{q} \) at a higher probability \( \Pi_s(\overline{q}) - \Pi_s(q) > 0 \). That is,

\[
P_s(\overline{q}) \mathbb{E}_t(\pi^*(t, \overline{q})) - P_s(q) \mathbb{E}_t(\pi^*(t, q)) > \overline{q}(\Pi_s(\overline{q}) - \Pi_s(q)) > q(\Pi_s(\overline{q}) - \Pi_s(q)).
\]

However, this expected payment difference is large enough to compensate for losing the item \( q \) at a higher probability. Thus, truthful report is not incentive compatible for the seller with quality \( q \).

5 The Irregular Case

In this section, we attend to the irregular case. Recall the definition of the break-even type

\[
t(q) \triangleq \sup\{ t \in T | \phi_b(t, q) \leq \phi_s(q) \}.
\]

Here \( t \) and \( q \) can be arbitrarily coupled in the virtual value \( \phi_b \), which makes it hard to pin down the break-even type. To improve the tractability, we make the following assumption.

**Assumption 1.** The buyer’s value \( v(t, q) \) is multiplicative separable in \( t \) and \( q \), i.e., \( v(t, q) = \alpha_1(q) \chi(t) + \alpha_2(q) \).

Since \( v(t, q) \) is continuous and increasing in \( t \), here we have \( \chi(\cdot) \) being continuous and \( \alpha_1(\cdot) > 0 \). Without loss of generality, we can normalize \( \chi(t) = t \). Then we have \( v(t, q) = \alpha_1(q) \chi(t) + \alpha_2(q) \).

\[ (\pi, P_b, P_s) \] by shifting up \( P_b \) under constraint (12) to the point where \( U_b(t_1) = 0 \). It generates higher revenue.
\( \alpha_1(q)t + \alpha_2(q) \). Recall the standard definition of the virtual type is

\[
\psi(t) \triangleq t - \frac{1 - F(t)}{f(t)}.
\]

The trading condition in Proposition 2 simplifies to

\[
\phi_b(t, q) = \alpha_1(q)\psi(t) + \alpha_2(q) \geq \phi_s(q),
\]

where \( \psi(t) \geq T(q) \)

where \( T(q) \triangleq \frac{\phi_s(q) - \alpha_2(q)}{\alpha_1(q)} \) is the buyer’s virtual type when his virtual value equals the seller’s virtual cost. We thus call \( T(q) \) the break-even virtual type. Then a problem is regular if \( \psi(t) \) and \( T(q) \) are non-decreasing.

After decoupling \( t \) and \( q \), even if the regularity condition fails, we can iron \( \psi(t) \) and \( T(q) \) separately. We recall the ironing procedure.

**Definition 5.** Ironing function \( \psi(t) \).

1. For any \( \omega \in [0, 1] \), let

\[
h_b(w) \triangleq \psi(F^{-1}(w)),
\]

where \( F^{-1}(w) \) is the inverse function of \( F(t) \). Note that \( F(t) \) is continuous and strictly increasing since we assume that the density function \( f(t) \) is always strictly positive. Thus the inverse function \( F^{-1} \) is also continuous and strictly increasing.

2. Let \( H_b : [0, 1] \mapsto \mathbb{R} \) be the integral of \( h_b(w) \):

\[
H_b(w) \triangleq \int_0^w h_b(r) \, dr.
\]

3. Let \( L_b : [0, 1] \mapsto \mathbb{R} \) be the convex envelope of the function \( H_b \):

\[
L_b(w) \triangleq \min \{ \lambda H_b(w_1) + (1 - \lambda) H_b(w_2) \},
\]
where $\lambda, w_1, w_2 \in [0, 1]$ and $\lambda w_1 + (1 - \lambda)w_2 = w$.

4. Define $l_b : [0, 1] \mapsto \mathbb{R}$ such that

$$l_b(w) \triangleq L'_b(w).$$

5. Then we obtain the ironed function $\bar{\psi}$

$$\bar{\psi}(t) \triangleq l_b(F(t))$$

whenever this derivative is defined, and we extend $l$ to all of $[0, 1]$ by right-continuity.

The mediator’s revenue function can be rewritten as

$$\int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \alpha_1(q) [\psi(t) - \mathcal{T}(q)] \, dt \, dq$$

$$= \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \alpha_1(q) [\bar{\psi}(t) - \mathcal{T}(q)] \, dt \, dq$$

$$+ \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \alpha_1(q) [h_b(F(t)) - l_b(F(t))] \, dt \, dq$$

since $h_b(F(t)) = \psi(t), l_b(F(t)) = \bar{\psi}(t)$. We can simplify the second term above via integration by parts:

$$\int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \alpha_1(q) [h_b(F(t)) - l_b(F(t))] \, dt \, dq$$

$$= \int_{t_1}^{t_2} [h_b(F(t)) - l_b(F(t))] \Pi_b(t; \pi) \, dF(t)$$

$$= [H_b(F(t)) - L_b(F(t))] \Pi_b(t; \pi) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} [H_b(F(t)) - L_b(F(t))] \, d\Pi_b(t; \pi).$$

By definition, $L_b$ is the convex envelope of $H_b$, then we have $L_b(0) = H_b(0)$ and $L_b(1) = H_b(1)$. 

17
Therefore, the first term is 0, and the revenue function becomes:

$$\text{Rev}(\pi, P_b, P_s) = \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[\tilde{\psi}(t) - T(q)] \, dt \, dq$$

(8)

$$= \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[\tilde{\psi}(t) - \bar{T}(q)] \, dt \, dq$$

$$- \int_{t_1}^{t_2} [H_b(F(t)) - L_b(F(t))] \, d\Pi_b(t; \pi).$$

To deal with function $T(q)$, the classical procedure no longer works. Instead, we modify the standard ironing by changing the definition of $w$ to the following:

$$w(q) = \int_{q_1}^{q} \alpha_1(r)g(r) \, dr.$$

Note that $\alpha_1(q) > 0$ and $g(q) > 0$ for all $q \in [q_1, q_2]$. Thus, $w(q)$ is a strictly increasing function, and has an inverse function $w^{-1} : [0, \bar{w}] \mapsto [q_1, q_2]$, where $\bar{w} = \int_{q_1}^{q_2} \alpha_1(r)g(r) \, dr$.

When ironing function $T(q)$, we similarly define four functions $h_s(w)$, $H_s(w)$, $L_s(w)$ and $l_s(w)$ with domain $[0, \bar{w}]$. Let the ironed function be $\bar{T}(q) \triangleq l_s(w(q))$.

The first term in the revenue (8) can be written as

$$\int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[\tilde{\psi}(t) - \bar{T}(q)] \, dt \, dq$$

$$= \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[\tilde{\psi}(t) - \bar{T}(q)] \, dt \, dq$$

$$+ \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[l_s(w(q)) - h_s(w(q))] \, dt \, dq$$

due to $h_s(w(q)) = \bar{T}(q)$ and $l_s(w(q)) = \bar{T}(q)$. We can simplify the second term of the right-hand side as follows:

$$\int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)g(q)\alpha_1(q)[l_s(w(q)) - h_s(w(q))] \, dt \, dq$$

$$= \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q)f(t)[l_s(w(q)) - h_s(w(q))] \, dt \, dw(q)$$

$$= \int_{q_1}^{q_2} [l_s(w(q)) - h_s(w(q))] \Pi_s(q; \pi) \, dw(q)$$

$$= [L_s(w(q)) - H_s(w(q))] \Pi_s(q; \pi) \int_{q_1}^{q_2} [L_s(w(q)) - H_s(w(q))] \, d\Pi_s(q; \pi)$$
\( L_s(0) = H_s(0) \) and \( L_s(\bar{w}) = H_s(\bar{w}) \) since \( L_s \) is the convex envelope of \( H_s \). Thus the first term is equal to 0. Overall, the revenue function of the mediator can be written as

\[
\text{Rev}(\pi, P_b, P_s) = \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \alpha_1(q) [\bar{\psi}(t) - \bar{T}(q)] \, dt \, dq - \int_{t_1}^{t_2} [H_b(F(t)) - L_b(F(t))] \, d\Pi_b(t; \pi) - \int_{q_1}^{q_2} [L_s(W(q)) - H_s(W(q))] \, d\Pi_s(q; \pi).
\]

(9)

Now we present the revenue-maximizing mechanism for the general case, which also has a threshold structure.

**Proposition 3.** Suppose assumption 1 holds. The direct mechanism

\[
\pi^*(t, q) = \begin{cases} 
1 & \text{if } \bar{\psi}(t) \geq \bar{T}(q) \\
0 & \text{otherwise.}
\end{cases}
\]

with the same payment function in Proposition 2 is feasible and maximizes revenue.

## 6 Conclusion

We study the problem of designing revenue-maximizing mechanisms for an uninformed mediator who possesses communication devices. For tractability, we assume that the buyer’s valuation is multiplicatively separable in his type and the quality. One direction for future research is to allow players’ valuations to be arbitrarily correlated.
References


Appendix

A  Extensive-Form Game Induced by a Mechanism

A mechanism induces a finite extensive-form game. As the mediator has commitment power, the induced game is played between the seller and the buyer. Given a mechanism, we can view the mediator’s actions as chance moves. Formally, given a mechanism, a corresponding two-player extensive-form game can be defined as a tuple $\Gamma = \langle N, T, H, Z, H, A, I, u \rangle$, where

- $N = \{b, s\}$ is the set of players;
- $T$ is a game tree, obtained by adding a nature node to the mechanism. Nature chooses the types of players at the beginning of the game;
- $H$ is a set of non-terminal nodes;
- $Z$ is a set of terminal nodes;
- $H = \{H_0, H_s, H_b\}$ is a partition of $H$, where $H_0$ is the set of chance move nodes, and $H_s$ and $H_b$ are the sets of nodes where the seller and the buyer moves, respectively;
- $A$ is a function that maps a non-terminal node $h \in H$ to a set of available actions $A(h)$;
- $I = \{I_i\}_{i \in N}$ is a collection of information partitions, where each $I_i$ is a partition of $H_i$. An element $I \in I_i$ is called an information set of player $i$. Every information set $I$ satisfies $A(h) = A(h'), \forall h, h' \in I$. Thus for any information set $I \in I_i$, we denote by $A(I)$ the set $A(h)$ for all $h \in I$;
- $u = (u_i)_{i \in N}$ is a collection of payoff functions, where each $u_i : Z \mapsto \mathbb{R}$ is the payoff function of player $i$.

In such a game tree, any node in $T$ can be uniquely determined by the path from the root to the node itself. Such a path is also called the history of the node. Thus from now on, we abuse notation and use $h$ or $z$ to denote both a node and its history.
**Solution Concept.** Our solution concept is Perfect Bayesian Equilibrium (PBE).

A *strategy* $S_i(I)$ of player $i$ is a function that assigns a probability distribution over $A(I)$ to each information set $I \in \mathcal{I}_i$. Let $\mathcal{S}_i$ denote the set of all possible strategies of player $i$. Given a strategy profile $S = (S_i)_{i \in N}$, one can easily calculate the probability $\text{Prob}(z|S)$ of reaching a terminal node $z$ using the structure of the game tree.

A *belief* is a function $\mu : H \mapsto [0, 1]$ that assigns to each node $h \in H$ a probability such that the probabilities of the node in any information set sum up to 1, i.e., $\sum_{h \in I} \mu(h) = 1, \forall I \in \mathcal{I}_i$, for all players $i \in N$. $\mu(h)$ can be interpreted as the probability that the player believes they are at node $h$ when $I$ is reached.

An *assessment* is a pair $(S, \mu)$, where $S$ is a strategy profile and $\mu$ is a belief. Let $Z_I$ be the set of all terminal nodes $z$ reachable from some nodes in $I$, and $\text{Prob}(z|S, \mu, I)$ the conditional probability of reaching node $z \in Z_I$ given an assessment $(S, \mu)$ and information set $I \in \mathcal{I}_i$. Then the conditional expected payoff of player $i$ for $(S, \mu)$ at information set $I$ is defined as

$$U_{i,I}(S|\mu) = \sum_{z \in Z_I} \text{Prob}(z|S, \mu, I) u_i(z).$$

**B Omitted Proofs**

Proving Proposition 2 requires several steps of analysis. The first lemma characterizes all feasible mechanisms.

**Lemma 1.** A mechanism $(\pi, P_b, P_s)$ is feasible if and only if it satisfies the following constraints:

1. $Edv(t, x; \pi)$ is monotone non-decreasing in $t$. \hspace{1cm} (10)
2. $\Pi_s(q; \pi)$ is monotone non-increasing in $q$. \hspace{1cm} (11)
3. $U_b(t) = U_b(t_1) + \int_{t_1}^{t} \Pi_b(x; \pi) \, dx$ \hspace{1cm} (12)
4. $U_s(q) = U_s(q_1) - \int_{q_1}^{q} \Pi_s(x; \pi) \, dx$ \hspace{1cm} (13)
5. $U_b(t_1) \geq 0$ \hspace{1cm} (14)
6. $U_s(q_2) \geq 0$ \hspace{1cm} (15)
where $Edv(t, x; \pi) \triangleq \int_{q \in Q} \frac{\partial v(x, q)}{\partial x} \pi(t, q) g(q) \, dq$.

**Proof of Lemma 1.** It suffices to prove the case of the buyer.

We first prove the necessity. Manipulating the buyer’s IC constraint (3) yields

$$U_b(t) \geq U_b(t') + \int_{q \in Q} \pi(t', q)[v(t, q) - v(t', q)]g(q) \, dq$$

$$= U_b(t') + \int_{q \in Q} \pi(t', q)g(q) \, dq \int_{x=\pi}^{t} \frac{\partial v(x, q)}{\partial x} \, dx.$$  

The above inequality is equivalent to

$$U_b(t) - U_b(t') \geq \int_{q \in Q} \pi(t', q)g(q) \, dq \int_{x=\pi}^{t} \frac{\partial v(x, q)}{\partial x} \, dx. \quad (16)$$

The above equation should hold for any $t$ and $t'$. As a result, if we switch $t$ and $t'$, we should have

$$U_b(t') - U_b(t) \geq \int_{q \in Q} \pi(t, q)g(q) \, dq \int_{x=\pi}^{t'} \frac{\partial v(x, q)}{\partial x} \, dx. \quad (17)$$

Combining Equation (16) and (17) gives

$$\int_{q \in Q} \pi(t', q)g(q) \, dq \int_{x=\pi}^{t} \frac{\partial v(x, q)}{\partial x} \, dx \leq U_b(t) - U_b(t') \leq \int_{q \in Q} \pi(t, q)g(q) \, dq \int_{x=\pi}^{t'} \frac{\partial v(x, q)}{\partial x} \, dx. \quad (18)$$

When $t > t'$, we can divide the above inequality by $t - t'$

$$\int_{q \in Q} \pi(t', q)g(q) \, dq \int_{x=\pi}^{t} \frac{\partial v(x, q)}{\partial x} \, dx \leq \frac{U_b(t) - U_b(t')}{t - t'} \leq \int_{q \in Q} \pi(t, q)g(q) \, dq \int_{x=\pi}^{t'} \frac{\partial v(x, q)}{\partial x} \, dx, \quad (19)$$

Letting $t' \to t$, we have $\frac{\int_{x=\pi}^{t} \frac{\partial v(x, q)}{\partial x} \, dx}{t - t'} \to \frac{\partial v(t, q)}{\partial t}$ and the above inequalities become

$$\frac{dU_b(t)}{dt} = \int_{q \in Q} \frac{\partial v(t, q)}{\partial t} \pi(t, q)g(q) \, dq = \Pi_b(t; \pi). \quad (20)$$

The above equation still holds if $t < t'$. Therefore, Equation (12) follows. Adding Equation
(16) and (17) gives
\[
\int_t^{t'} dx [Edv(t, x; \pi) - Edv(t', x; \pi)] \geq 0
\]
which implies constraint (10). Moreover, the IR constraint implies \( U_b(t_1) \geq 0 \), which yields Equation (14).

Now we show sufficiency. By constraints (10) and (12), we have
\[
U_b(t) - U_b(t') = \int_t^{t'} \Pi_b(x; \pi) \, dx = \int_t^{t'} Edv(x, x; \pi) \, dx \geq \int_t^{t'} Edv(t', x; \pi) \, dx.
\]
Rearrange to obtain
\[
U_b(t) \geq U_b(t') + \int_{q \in Q} \pi(t', q) g(q) \int_{x=t'}^{t} \frac{\partial v(x, q)}{\partial x} \, dx.
\]
which is equivalent to the buyer’s IC constraint (3). Given \( \Pi_b(:, \pi) \geq 0 \), constraint (12) and (14) imply \( U_b(t) \geq 0 \) for any \( t \), which is the IR constraint.

Then we can rewrite the revenue of the mechanism.

**Lemma 2.** The expected revenue of any feasible mechanism \((\pi, P_b, P_s)\) is

\[
\text{Rev}(\pi, P_b, P_s) = \int_{t_1}^{t_2} \int_{q_1}^{q_2} \pi(t, q) f(t) g(q) [\phi_b(t, q) - \phi_s(q)] \, dt dq - U_b(t_1) - U_s(q_2).
\]

**Proof of Lemma 2.** For any feasible mechanism \((\pi, P_b, P_s)\), the revenue of the mediator can be written as

\[
\text{Rev}(\pi, P_b, P_s) = \int_{t \in T} f(t) \int_{q \in Q} g(q) \pi(t, q) P_b(t) \, dq \, dt - \int_{q \in Q} g(q) \int_{t \in T} f(t) \pi(t, q) P_s(q) \, dt \, dq.
\]
By (1), the first term can be written as

\[
\int_{t_1}^{t_2} f(t) \int_{q \in Q} g(q) \pi(t, q) P_b(t) \, dq \, dt \\
= \int_{t_1}^{t_2} f(t) \left[ \int_{q \in Q} \pi(t, q) v(t, q) g(q) \, dq - U_b(t) \right] \, dt,
\]

where the second term, \( \int_{t_1}^{t_2} f(t) U_b(t) \, dt \), can be expanded

\[
\int_{t_1}^{t_2} f(t) U_b(t) \, dt \\
= \int_{t_1}^{t_2} f(t) \left[ \int_{t_1}^{t} \Pi_b(x; \pi) \, dx + U_b(t_1) \right] \, dt \\
= U_b(t_1) + \int_{t_1}^{t_2} \int_{t_1}^{t} f(t) \Pi_b(x; \pi) \, dx \, dt \\
= U_b(t_1) + \int_{t_1}^{t_2} \int_{t_1}^{t} f(t) \Pi_b(x; \pi) \, dt \, dx \\
= U_b(t_1) + \int_{t_1}^{t_2} [1 - F(x)] \Pi_b(x; \pi) \, dx \\
= U_b(t_1) + \int_{q \in Q} g(q) \left[ \int_{t_1}^{t_2} [1 - F(t)] \frac{\partial v(t, q)}{\partial t} \pi(t, q) \, dt \right] \, dq.
\]

Plugging the term back into the above equation and switching the order of integral

\[
\int_{t_1}^{t_2} f(t) \int_{q \in Q} g(q) \pi(t, q) P_b(t) \, dq \, dt \\
= \int_{t_1}^{t_2} f(t) \left[ \int_{q \in Q} \pi(t, q) v(t, q) g(q) \, dq - \int_{q \in Q} \frac{1 - F(t)}{f(t)} \frac{\partial v(t, q)}{\partial t} \pi(t, q) g(q) \, dq \right] \, dt - U_b(t_1), \\
= \int_{q \in Q} g(q) \left[ \int_{t_1}^{t_2} f(t) \pi(t, q) \phi_b(t, q) \, dt \right] \, dq - U_b(t_1).
\]

The second term of Equation (22) is about the seller. According to Equation (2), it can be written as

\[
\int_{q_1}^{q_2} g(q) \int_{t \in T} f(t) \pi(t, q) P_s(q) \, dt \, dq \\
= \int_{q_1}^{q_2} g(q) \left[ \int_{t \in T} \pi(t, q) r(q) f(t) \, dt + U_s(q) \right] \, dq,
\]

(23)
where we can rewrite the second term as

\[
\int_{q_1}^{q_2} g(q) U_s(q) \, dq
= \int_{q_1}^{q_2} g(q) \left[ \int_q^{q_2} \Pi_s(x; \pi) \, dx + U_s(q_2) \right] \, dq
= \int_{q_1}^{q_2} g(q) \int_q^{q_2} \Pi_s(x; \pi) \, dx \, dq + U_s(q_2)
= \int_{q_1}^{q_2} G(x) \Pi_s(x; \pi) \, dx + U_s(q_2)
= \int_{q_1}^{q_2} f(t) \int_{q_1}^{q_2} G(q) \pi(t, q) \, dq \, dt + U_s(q_2)
\]

Putting everything back to Equation (23) yields

\[
\int_{q_1}^{q_2} g(q) \int_{t \in T} f(t) \pi(t, q) P_s(q) \, dt \, dq
= \int_{t \in T} f(t) \left[ \int_{q_1}^{q_2} g(q) \pi(t, q) \left( q + \frac{G(q)}{g(q)} \right) \, dq \right] \, dt + U_s(q_2)
= \int_{t \in T} f(t) \left[ \int_{q_1}^{q_2} g(q) \pi(t, q) \phi_s(q) \, dq \right] \, dt + U_s(q_2),
\]

By combining both two terms of Equation (22),

\[
Rev(\pi, P_b, P_s)
= \int_{q_1}^{q_2} \int_{t_1}^{t_2} \pi(t, q) f(t) g(q) \left[ \phi_b(t, q) - \phi_s(q) \right] \, dt \, dq - U_b(t_1) - U_s(q_2).
\]

Then we show that the mechanism described in Proposition 2 is feasible.

**Lemma 3.** For a regular problem, the mechanism \((\pi^*, P^*_b, P^*_s)\) defined in Proposition 2 is feasible.

**Proof of Lemma 3.** Observe that if a problem instance is regular, the recommendation scheme \(\pi^*(t, q)\) is non-decreasing in \(t\) and non-increasing in \(q\). This implies that \(\Pi_b(t; \pi)\) is non-
decreasing in $t$ and $\Pi_s(q; \pi)$ is non-increasing in $q$.

Recall that

$$P_b^*(t) = \mathbb{E}_q(v(t, q) | \pi^*(t, q) = 1) - \frac{1}{\mathbb{E}_q(\pi^*(t, q))} \int_{t_1}^t \Pi_b(x; \pi^*)dx,$$

$$P_s^*(q) = \mathbb{E}_t(r(q) | \pi^*(t, q) = 1) + \frac{1}{\mathbb{E}_t(\pi^*(t, q))} \int_q^{q_2} \Pi_s(x; \pi^*)dx.$$

By definition, the buyer’s payoff is

$$U_b(t) = \int_{q \in Q} \pi^*(t, q)[v(t, q) - P_b^*(t)]g(q) dq$$

$$= \int_{t_1}^t \int_{q \in Q} \frac{\partial v(t, q)}{\partial t} \pi^*(x, q)g(q) dq dx.$$

This means $U_b(t_1) = 0$ and that

$$\frac{dU_b(t)}{dt} = \int_{q \in Q} \frac{\partial v(t, q)}{\partial t} \pi^*(t, q)g(q) dq = \Pi_b(t; \pi^*).$$

Therefore, Equation (12) follows.

Similarly, by definition, the seller’s payoff is

$$U_s(q) = \int_{t \in T} \pi^*(t, q)[P_s^*(q) - r(q)]f(t) dt$$

$$= \int_q^{q_2} \int_{t \in T} \pi^*(t, x)f(t) dt dx.$$

It follows that $U_s(q_2) = 0$ and that

$$\frac{dU_s(q)}{dq} = -\int_{t \in T} \pi^*(t, q)f(t) dt = -\Pi_s(q; \pi^*)$$

implying Equation (13).

Proof of Proposition 2. The mechanism $(\pi^*, P_b^*, P_s^*)$ maximizes the revenue (21). The trading rule $\pi^*$ by design maximizes the first term. Moreover, the construction of the payment function ensures $U_b(t_1) = U_s(q_2) = 0$. □
Proof of Proposition 3. We show that the mechanism \((\pi^*, P^*_b, P^*_s)\) maximizes all the terms in Equation (9) simultaneously. The first term is maximized by \(\pi^*\), as \(\pi^*(t, q) = 1\) only when \(\bar{\varphi}_b(t) - \bar{T}(q) \geq 0\). Moreover, \((\pi^*, P^*_b, P^*_s)\) satisfies \(U_b(t_1) = 0\) and \(U_s(q_2) = 0\), indicating that it also maximizes the last two terms since Lemma 1 implies that \(U_b(t_1) \geq 0\) and \(U_s(q_2) \geq 0\) hold for all feasible mechanisms.

As for the second term, it is worth noting that \(H_b(F(t)) - L_b(F(t)) \geq 0\) since \(L_b(w)\) is the convex envelope of \(H_b(w)\). Additionally, based on Lemma 1, we know that \(\text{d} \Pi_b(t; \pi) \geq 0\), which means this term is always non-negative. Therefore, in order to show that the second term is maximized, it suffices to prove that this term equals 0. Actually, it is only necessary to consider cases where \(H_b(F(t)) - L_b(F(t)) > 0\). In such cases, \(t\) must fall within an ironed interval \(I\), thus function \(L_b(w)\) is linear in the interval \(I\). This implies \(l_b(w) = \tilde{\varphi}_b(t)\) is a constant and thus \(\Pi_b(t; \pi) = \int_{q: \bar{T}(q) \leq \bar{\varphi}_b(t)} \alpha_1(q) g(q) \, dq\) is also a constant in the interval \(I\), leading to \(\text{d} \Pi_b(t; \pi^*) = 0\).

Similarly, for the third term, we observe that \(L_s(w(q)) - H_s(w(q)) \leq 0\) since \(L_s(w)\) is a convex envelope of \(H_s(w)\). According to Lemma 1, \(\text{d} \Pi_s(q; \pi) \leq 0\) holds for all feasible mechanisms. Consequently, this term is also non-negative. We show that in the optima mechanism \((\pi^*, P^*_b, P^*_s)\), this term equals to 0. Consider cases where \(L_s(w(q)) - H_s(w(q)) < 0\). In such cases, \(q\) must lie in an ironed interval \(I\) and thus \(L_s(w)\) is linear in the ironing interval. This implies \(l_s(w) = \bar{T}(q)\) is a constant. Then \(\Pi_s(q; \pi^*) = \int_{t: \bar{\varphi}_b(t) \geq \bar{T}(q)} f(t) \, dt\) is also a constant in the interval \(I\), leading to \(\text{d} \Pi_s(q; \pi^*) = 0\).

In conclusion, the mechanism \((\pi^*, P^*_b, P^*_s)\) optimizes all terms in the Equation (9) simultaneously, proving it to be a revenue-maximizing feasible mechanism. □