Dominant Equilibria in Signaling Games

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Abstract

We study dominant separating equilibria in quadratic signaling games. We show the sender has a linear incentive compatible separating strategy if and only if the game has a non-negative discriminant. We relax the common belief monotonicity assumption, and show that in non-negative discriminant games there exists a unique dominant separating strategy that is continuous and differentiable. We derive sufficient and necessary conditions for this strategy to be linear. We fully characterize the dominant separating perfect Bayesian equilibrium and establish its existence and uniqueness. We apply these results to confirm the dominance of linear separating equilibria in some classic examples, and show that, in other classic examples, there exist previously unknown non-linear dominant equilibria.

Keywords: Signaling; Separating Equilibrium; Linear Strategy.

JEL. C73. D82. D83.

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Signaling games play an important role in many areas of social science, proving insights on issues such as education (Spence, 1973), limit pricing (Milgrom and Roberts, 1982), leadership (Hermalin, 1998), insurance (Rothschild and Stiglitz, 1976). In signaling games, a privately informed sender strategically takes an action to influence the action of an uninformed receiver. A particularly important case is when the equilibrium strategy is separating and thus information is transmitted perfectly. For tractability, the literature imposes some preference restrictions, including the widely-assumed belief monotonicity\(^1\) (e.g., Spence, 1973; Mailath, 1987; Roddie, 2011; Mailath and von Thadden, 2013).\(^2\) Another type of restrictions is on what strategy the sender can take. Some recent studies restrict attention to linear equilibria.\(^3\) An important but unexplored question is whether these restrictions have economic significance: does a more general theory of signaling games provide qualitatively different predictions?

We study signaling games without imposing any of the above restrictions. We focus on quadratic signaling games, where the sender has an additive quadratic preference. This setup is mathematically general and nests a broad class of models. We explore the dominant separating strategy; it is the strategy that maximizes the sender’s payoff at every state. We are interested in the dominant strategy since it is the sender’s optimal strategy. We endogenize linearity, continuity, monotonicity, and differentiability when analyzing the dominant strategy. To compare linear versus non-linear strategies, we first analyze the existence of a linear strategy.

We introduce a quadratic game characteristic called discriminant, and our first result shows that there exists a linear incentive compatible strategy if and only if the game has a non-negative discriminant (Theorem 1). Then we analyze non-negative discriminant games.\(^4\) We first show that they have a unique dominant strategy, and derive sufficient and necessary conditions for this strategy to be linear (Theorem 2).

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\(^1\)The sender always prefers a higher belief, regardless of the state and his action.

\(^2\)There are several other restrictions in the literature. For example, Kartik et al. (2007) propose a direction condition: given correct belief, taking a higher action and inducing a higher belief affect the sender’s payoff in the same direction. In the same vein, Mailath and von Thadden (2013) assume action monotonicity: under correct belief, the sender always prefers a higher action. Kartik et al. (2007) adopt the weaker assumption of local belief monotonicity. We relax all these restrictions in our model.

\(^3\)See e.g., Argenziano and Bonatti (2021); Ball (2021); Bonatti and Cisternas (2019). A similar restriction is monotone equilibria (e.g., Kartik, 2009). Liu and Pei (2020) provide sufficient conditions under which all equilibria are monotone.

\(^4\)We discuss negative discriminant games in the online appendix.
Moreover, we show that monotonicity, continuity, and differentiability arise endogenously in the dominant strategy. Second, we establish existence and uniqueness of the dominant perfect Bayesian equilibrium (PBE), and show that in this PBE the sender takes the dominant strategy (Theorem 3). In addition to the existence of the dominant PBE, its uniqueness is non-trivial since we can no longer invoke the classic initial condition (Mailath, 1987). Third, we show that under local belief monotonicity, the dominant strategy satisfies a generalized initial condition. But when local belief monotonicity is violated, the dominant strategy features a novel value condition (Proposition 2).

Our results show that the common restriction on linear strategy is only partially justified, and that in general a wider class of behaviors is possible. On the other hand, assumptions as continuity, differentiability and monotonicity do arise endogenously and do not need to be imposed by the modeler. Furthermore, qualitatively different behaviors indeed appear once we relax belief monotonicity.

We illustrate these by applying our results to four examples in the literature. In Hermalin (1998) and in Argenziano and Bonatti (2021), the authors select the linear equilibrium. We show that the linear equilibrium is indeed dominant only in a subset of the parameter space. We exhaustively analyze the remainder of the space, where there exist previously unknown non-linear dominant equilibria. In a model of lying costs, Kartik et al. (2007) analyze the case where the informed sender’s preference is biased upward relative to the receiver’s and show that inflated language arises. Relaxing belief monotonicity allows the sender’s preference to be biased upward for some states but downward for other states. We show that the language is inflated if and only if the sender is biased upward (Proposition 4). In the fourth example, due to Aghion et al. (2004), we formalize the idea of transferable control in a signaling game with a continuous state, and show that the (agent) dominant equilibrium is also optimal for the principal (Proposition 5).

Why do we relax belief monotonicity? Mailath (1987) discusses the differentiability of separating strategies in signaling games. In that paper, belief monotonicity is needed to provide two additional conditions. The first is an initial value condition. The second is a single crossing condition. Each of these two (combined with regularity conditions) implies differentiability of separating strategies. The initial value

\[^5\text{Ramey (1996) shows that with multiple signals and a continuum of states, D1 refinement selects the dominant separating equilibrium under strong single crossing condition. But our model does not impose the strong single crossing condition.}\]

\[^6\text{Given belief monotonicity, Roddie (2011) finds that the dominant separating strategy satisfies the initial condition. As we do not impose belief monotonicity, we can generalize this result.}\]
condition and incentive compatibility together can uniquely pin down an equilibrium. Nevertheless, belief monotonicity is quite restrictive. For example, in the standard cheap talk model of Crawford and Sobel (1982), whenever communication is informative, the preference bias between the sender and the receiver is relatively small so that belief monotonicity fails. Thus studying signaling games without belief monotonicity is a natural step forward.

2 The Model

We define quadratic signaling games as follows. Let $\theta \in \Theta$ denote the state of the world where $\Theta = [m,M]$ is a bounded closed interval in $\mathbb{R}$. A sender (he) is privately informed about the underlying state $\theta$ and takes an action $a_1 \in \mathbb{R}$. Based on $a_1$, a receiver (she) chooses an action $a_2 \in \mathbb{R}$. The receiver’s preference is given by $u^R(a_1, \cdot, \theta)$. We assume $u^R(a_1, \cdot, \theta)$ has a single peak at $a_2 = a^R(\theta)$ that is continuous and strictly increasing in $\theta$. Without loss of generality, we normalize $a^R = \theta$.\footnote{Given state $\tilde{\theta}$ and $a^R(\tilde{\theta})$, we can redefine the state $\theta \triangleq a^R(\tilde{\theta})$.} The sender’s preference is given by

$$U^S(a_1, a_2, \theta) = -(a_1 - a^S_1(\theta))^2 - \delta(a_2 - a^S_2(\theta))^2,$$

where $a^S_1 : \Theta \to \mathbb{R}$ and $a^S_2 : \Theta \to \mathbb{R}$ denote the sender’s ideal actions for $a_1$ and $a_2$, respectively. We assume they are linear in the state: $a^S_1(\theta) = k_1 \theta + b_1$, $a^S_2(\theta) = k_2 \theta + b_2$ with $k_1 \neq 0$. $\delta \geq 0$\footnote{Our results apply to the case where $\delta < 0$ as well. But one needs to additionally check a second order condition. We provide an example in Section 6.2.} captures the sensitivity of the sender’s payoff to the receiver’s action. The quadratic form succinctly captures some key properties of concave preferences and allows for closed form solutions.

Quadratic signaling games include the games studied in Argenziano and Bonatti (2021) and Kartik et al. (2007). With slight mathematical manipulation, they also include the game in Hermalin (1998). We discuss these models in Section 6.

We provide a simple illustrating example. Consider a principal (she) who just acquired a firm. Without detailed knowledge about how the firm operates (the state of the firm), the principal can first delegate control to an executive (the agent, he) of the firm to utilize his knowledge. After learning from the agent, the principal reclaims control in the future. The principal is unable to commit to contingent transfers and cares about the firm’s profit. Thus, she prefers to implement a decision to match the
state $a^R = \theta$. Meanwhile, the agent prefers the firm to operate at a larger scale than necessary (Donaldson, 1984; Williamson, 1963) and wants to implement a decision $a^S_1 = a^S_2 = \theta + b$ where $b$ measures the preference bias. It is helpful to keep this example in mind. Next, we attend to the solution concept.

A pure strategy $a_1(\theta)$ is *separating* if $a_1(\theta) \neq a_1(\theta')$ for all $\theta, \theta' \in \Theta$. We restrict attention to separating strategies and consider separating perfect Bayesian equilibria (PBE). For the rest of the paper, a strategy/PBE refers to a pure separating strategy/PBE. Let $A_1 \triangleq \{a_1(\theta) | \theta \in \Theta\}$. A PBE consists of a strategy $a_1 : \Theta \to \mathbb{R}$ and the receiver’s belief $\hat{\theta} : \mathbb{R} \to \Theta$ such that:

1. Belief consistency: $\forall \theta \in \Theta$, the receiver’s belief $\hat{\theta}(a_1(\theta)) = \theta$;
2. Incentive compatibility (IC): $\forall \theta, \theta' \in \Theta$, $U^S(a_1(\theta), \theta, \theta) \geq U^S(a_1(\theta'), \theta', \theta)$;
3. On the equilibrium path: $\forall \theta \in \Theta$, $\forall a \notin A_1$, $U^S(a_1(\theta), \theta, \theta) \geq U^S(a, \hat{\theta}(a), \theta)$.

A *dominant strategy* maximizes the sender’s payoff at every state among all incentive compatible strategies. A *dominant PBE* maximizes the sender’s payoff at every state among all PBE.

In equilibrium, the receiver takes the action $a^R(\theta) = \theta$, which might differ from the sender’s ideal point $a^S_2(\theta) = k_2 \theta + b_2$. The distance between them is $a^S_2 - a^R = k_2 \theta + b_2 - \theta$. First, we define the *preference-aligned state* $\theta_0$ where this distance is zero:

$$\theta_0 = k_2 \theta_0 + b_2,$$

i.e., $\theta_0 \triangleq \frac{b_2}{1 - k_2}$. $\theta_0$ may or may not be in $\Theta$, depending on the parameter. Second, this distance as a function of the state $\theta$ increases at rate $(k_2 - 1)$. The larger the distance, the lower the sender’s payoff. Recall $\delta$ captures the sensitivity of the sender’s payoff to the receiver’s action. To measure the impact of a unit state increment $d\theta$ on the sender’s payoff, we define the *marginal impact* $r \triangleq \delta(k_2 - 1)$.

### 3 Preliminary Analysis

Similar to Mailath (1987), we can summarize the interaction between the sender and the receiver as a $C^2$ function:

$$V(\theta, \hat{\theta}, a_1) = -(a_1 - k_1 \theta - b_1)^2 - \delta(\hat{\theta} - k_2 \theta - b_2)^2,$$  \hspace{1cm} (1)\footnote{If $k_2 = 1$, we can define $\theta_0 = \infty$.}
which denotes the sender’s payoff from taking action $a_1$ when the true state is $\theta$ and the receiver infers $\hat{\theta}$. In the literature, the belief monotonicity assumption is

$$V_2(\theta, \hat{\theta}, a_1) \neq 0 \quad \forall (\theta, \hat{\theta}, a_1),$$

where subscripts on functions denote derivatives. That is, the sender always prefers a higher (lower) belief, regardless of the state and his action. Belief monotonicity is satisfied if and only if $k_2\theta + b_2 \notin \Theta$ for all $\theta \in \Theta$. Kartik et al. (2007) consider a weaker condition

$$V_2(\theta, \theta, a_1) \neq 0 \quad \forall (\theta, a_1)$$

which we call local belief monotonicity. That is, given a correct belief, the sender always prefers a slightly higher (lower) belief, regardless of his action. Local belief monotonicity is satisfied if and only if $\theta_0 \notin \Theta$.

As the starting point of our analysis, we reestablish a result of Mailath (1987) without imposing belief monotonicity.

**Lemma 1.** In every separating PBE with strategy $a_1$, if there exists an interval $(\underline{\theta}, \overline{\theta}) \subset \Theta$ such that $a_1(\theta)$ is continuous on $(\underline{\theta}, \overline{\theta})$, then for each $\theta \in (\underline{\theta}, \overline{\theta})$,

$$a_1'(\theta) = \frac{r\theta + \delta b_2}{a_1(\theta) - k_1\theta - b_1}$$

(2)

Next, we introduce a useful definition. We denote by $S(\theta) \triangleq a_1(\theta) - a_1^S(\theta) = a_1(\theta) - k_1\theta - b_1$ the equilibrium distortion, i.e., the distance between the sender’s action and his ideal point. If the sender is completely naive, he ignores the inferential impact of his action and myopically chooses $S(\theta) = 0$ for all $\theta$. By contrast, the strategic sender distorts his equilibrium behavior away from the naive benchmark. Consequently, $S(\theta)$ captures the equilibrium distortion in behavior.

We can thus rewrite the differential equation (2) as

$$S(\theta)[\frac{dS(\theta)}{d\theta} + k_1] = r\theta + \delta b_2$$

(3)

Solving for $a_1$ is equivalent to solving for $S$ and we can focus on equation (3) instead. Although Lemma 1 proves differentiability for any continuous IC strategy, an IC strategy is not necessarily continuous. For any discontinuous strategy, if the function $|S(\theta)|$ is discontinuous, so is the sender’s payoff $U^S = -S(\theta)^2 - \delta(\theta - k_2\theta - b_2)^2$, violating the indifference condition at the discontinuity. Thus, the function $|S(\theta)|$ must be continuous everywhere. This continuity simplifies the analysis for $|S(\theta)|$ and
we use this to generalize the Theorem 1 in Mailath (1987) without belief monotonicity.

**Proposition 1.** Any separating IC strategy has at most one discontinuity on $\Theta$, and where it is continuous, it is differentiable and satisfies (2). Furthermore, the jump at the discontinuity is of the same sign as $k_1$.

Notice that an IC separating strategy can be neither monotonic nor continuous. We provide an example in the proof of Proposition 5.

## 4 Main Results

The dominant separating equilibrium, also known as the Riley equilibrium, maximizes the sender’s payoff of all types. First, we are interested in the existence and uniqueness of this dominant PBE. Second, we strive to fully characterize the dominant PBE if it exists. Moreover, what is the dominant separating strategy? Is it monotonic, continuous? Another question of particular interest is whether the dominant strategy is linear.

Before analyzing when the dominant strategy is linear, we first need to know when a linear incentive compatible strategy exists. It turns out a linear incentive compatible strategy does not always exist. We show that its existence is determined by what we call the *discriminant* of the game: $\Delta \triangleq k_1^2 + 4r$.

**Theorem 1.** There exists a linear incentive compatible strategy if and only if $\Delta \geq 0$.

To illustrate the intuition behind this result, let us fix any linear strategy $S(\theta) = t\theta + l$, parametrized by $t$ and $l$. Then the marginal benefit of inducing a higher belief is

$$ \frac{dV(\theta, \hat{\theta}, a_1(\hat{\theta}))}{d\hat{\theta}} \bigg|_{\hat{\theta} = \theta} = -2[(t^2 + k_1t - r)(\theta - \theta_0) + l(t + k_1)]. $$

For $S(\theta)$ to be incentive compatible, this must vanish for every $\theta$. In particular, the coefficient of $\theta$ must vanish: $t^2 + k_1t - r = 0$. We call $\Delta$ the discriminant of the game, because it is the discriminant of this equation. When this discriminant is negative, there are no solutions, and it follows that the marginal benefit of inducing a higher strategy does not vanish for all $\theta$. Thus a linear strategy cannot be incentive compatible.

Since we want to compare linear versus non-linear strategies and see when widely-used linear strategies are indeed dominant, we focus on non-negative discriminant games in the main body of the paper. And we relegate the discussion of negative
discriminant games to the online appendix. There we establish similar results for non-linear strategies.

As a first step to solve the dominant PBE, we first characterize the dominant separating strategy.

**Theorem 2.** In non-negative discriminant quadratic games, there exists a unique dominant separating strategy. Moreover, it is

1. continuous, monotonic, and differentiable,
2. linear if and only if \( r > 0 \) and \( \theta_0 \in \Theta \).

Theorem 2 establishes the existence and uniqueness of the dominant strategy. Surprisingly, it guarantees appealing properties, including monotonicity and continuity. Furthermore, it delivers sufficient and necessarily condition for the dominant strategy to take a linear form. Nevertheless, this theorem remains silent regarding perfect Bayesian equilibrium. On top of incentive compatibility, PBE additionally requires that a sender of any type does not deviate off the equilibrium path. To address this issue, our next lemma constructs off-path beliefs for a broad class of separating strategies, including the dominant strategy, to keep all types on the equilibrium path.

**Lemma 2.** Let \( B \triangleq \{ a_1^S(\theta) | \theta \in \Theta \} \). If \( A_1^S \subset A_1 \), a continuous IC strategy \( a_1(\theta) \) and degenerate off-path belief

\[
\hat{\theta}(a_1) = \begin{cases} 
\bar{\theta}, & \text{if } a_1 < \min_{\theta \in \Theta} a_1(\theta) \\
\bar{\theta}, & \text{if } a_1 > \max_{\theta \in \Theta} a_1(\theta).
\end{cases}
\]

with any \( \bar{\theta} \in \arg \min_{\theta \in \Theta} a_1(\theta) \), \( \bar{\theta} \in \arg \max_{\theta \in \Theta} a_1(\theta) \) form a perfect Bayesian equilibrium.

Intuitively speaking, when \( A_1 \) contains all types’ bliss points, choosing any action outside \( A_1 \) is strictly worse than staying on the equilibrium path. Now we can prove a stronger conclusion than merely solving for the dominant PBE. We show that the sender takes the dominant strategy in the dominant PBE.

**Theorem 3.** In non-negative discriminant quadratic games, there exists a unique dominant PBE. Moreover, in this PBE, the sender takes the dominant strategy identified in Theorem 2.
Theorem 3 establishes the existence and uniqueness of the dominant PBE. Moreover, the sender takes the dominant separating strategy in this equilibrium. In fact, Theorem 2 and 3 apply to models beyond quadratic signaling games.\(^\text{10}\) As long as the sender’s action satisfies equation (2) whenever it is continuous, Theorem 2 and 3 apply to all the models satisfying the regularity conditions in Mailath (1987). We provide an example in Section 6.1.

We fully solve the dominant strategy’s closed form in the proof contained in the appendix. Here we provide a sketch of proof of both theorems to offer the main insights.

Recall the equilibrium distortion \(S(\theta)\) measures the distance between the sender’s action and his ideal point. As his loss increases in \(|S(\theta)|\), the dominant strategy is closest to the sender’s bliss point \(S = 0\) among all separating strategies. The solution of differential equation (3) relies crucially on the preference-aligned state \(\theta_0\). Recall it is the state where there is no conflict of interests. This feature leads \(\theta_0\) to be a singularity of the differential equation (3). In addition, how the integral curve behaves around the singularity depends on the marginal impact \(r\). We first plot the integral curve for \(r > 0\) in Figure 1. When \(r > 0\), there are two linear solutions \(S = w_1(\theta - \theta_0)\) and \(S = w_2(\theta - \theta_0)\). Without loss of generality, we let \(|w_1| \leq |w_2|\).

Suppose \(\theta_0 \notin \Theta\). Without loss of generality, we can only consider the case \(\theta_0 < m\). There is one integral curve crossing \((\theta, S) = (m, 0)\) and we let \(S^*\) denote its positive branch \((S^* \geq 0)\). By design, \(S^*\) is uniformly closest to \(S = 0\) among all positive

\(^{10}\)The proof of our two theorems only relies on Proposition 1. Theorem 1 in Mailath (1987) yields a similar result by imposing some regularity conditions.
Figure 2: Solutions for $r < 0$.

solutions of equation (3). By the Monotone Expansion Lemma (in the appendix), $S^*$ uniformly dominates all negative solutions of (3) as well. Since any discontinuous separating strategy jump upwards with $|S|$ being continuous, it is uniformly dominated by $S^*$. Therefore, $S^*$ is the dominant strategy, which is continuous, non-linear and unique. It is also monotonic, since $\frac{d\omega_1(\theta)}{d\theta} = \frac{r(\theta-\theta_0)}{S(\phi)} > 0$ by (3). Moreover, it supports a PBE by Lemma 2.

Suppose $\theta_0 \in \Theta$. All solutions for differential equation (3) defined on $\Theta$ are uniformly dominated by $S = w_1(\theta - \theta_0)$. Thus, the dominant strategy $S = w_1(\theta - \theta_0)$ is continuous, linear and unique. By the same argument above, it is monotonic and supports a PBE.

To avoid technical details, we briefly illustrate why linear strategies are not optimal for the case of $r < 0$. We plot the solutions in Figure 2. We can easily see that there are non-linear solutions uniformly closer to the sender’s bliss point $S = 0$. Therefore, the linear solutions are always sub-optimal.

**Remark 1.** In Online Appendix B.2, we investigate negative discriminant quadratic games. We find that when $\theta_0 \notin \Theta$, there exists a unique dominant PBE with non-linear, continuous, monotonic, and differentiable strategy. And when $\theta_0 \in \Theta$, there is no continuous incentive compatible separating strategy and moreover, a separating PBE may not exist.
5 Properties of The Dominant Strategy

In this section, we highlight some properties of the dominant strategy/PBE. First, given belief monotonicity, we illustrate how the dominant strategy rationalizes the classic initial condition of Mailath (1987). This condition requires \( S(\theta^w) = 0 \) where \( \theta^w \) is the worst type, i.e., the worst point belief the receiver can hold. For instance, if \( V_2(\theta, \hat{\theta}, a_1) > 0 \) for all \((\theta, \hat{\theta}, a_1), \theta^w = m \). We show this initial condition is implied by the dominant strategy, given belief monotonicity. Second, we generalize this result as we gradually relax belief monotonicity.

The initial condition is justified by the sequentiality of many games. That is, since \( \theta^w \) is the worst belief, if \( S(\theta^w) \neq 0 \), a deviation by \( S(\theta^w) \neq 0 \) to \( S(\theta^w) = 0 \) cannot be credibly punished in equilibrium. However, this sequentiality is a pure equilibrium refinement condition, which is completely silent about why the sender chooses so in the first place.

To connect to Mailath (1987), consider a setting satisfying belief monotonicity. Without loss of generality, we take \( k_2 \theta + b_2 > M \) for all \( \theta \) for an illustration. In this setting, \( V_2(\theta, \hat{\theta}, a_1) > 0 \) for all \((\theta, \hat{\theta}, a_1)\). Thus, the worst type \( \theta^w = m \). Surprisingly, the dominant strategy features \( S(m) = 0 \). That is, the traditional initial condition is implied by the optimality of the sender’s strategy. Whenever the sender chooses the strategy out of his own interests, we do not need to impose the initial condition which comes as part of the sender-dominant strategy.

To generalize this result, let us modify the example above. Without loss of generality, consider \( k_2 > 1 \) and \( k_2 m + b_2 \in \Theta \) where belief monotonicity fails. The dominant strategy in this setting still features \( S(m) = 0 \). But how do we extrapolate the intuition above? The answer lies in local belief monotonicity. In this example, since \( V_2(\theta, \theta, a_1) > 0 \) for all \((\theta, a_1)\), we can similarly define a “generalized worst type” \( \theta^w = m \), and the dominant strategy still implies \( S(\theta^w) = 0 \).

What happens if we lose local belief monotonicity? Local belief monotonicity is violated if and only if \( \theta_0 \in \Theta \). In this situation, the dominant strategy features \( S(\theta_0) = 0 \). Yet it is inappropriate to view this as an initial condition since \( \theta_0 \) is a singularity of the differential equation. At \( \theta = \theta_0 \), in the dominant PBE, the sender obtains the maximum payoff possible \( U^S = 0 \). That is, the sender achieves his bliss points for both \( a_1 \) and \( a_2 \). Since preferences are aligned at \( \theta_0 \), the sender willingly reveals his type based on which the receiver takes an action most beneficial for both of them.

We summarize the discussion above. Let \( \theta^w \) denote the generalized worst type...
when local belief monotonicity holds. Formally, \( \theta^w = m \) if \( V_2(\theta, \theta, a_1) > 0 \) for all \((\theta, a_1)\); \( \theta^w = M \) if \( V_2(\theta, \theta, a_1) < 0 \) for all \((\theta, a_1)\).

**Proposition 2.** In non-negative discriminant quadratic games, for the dominant separating strategy, \( S(\theta^w) = 0 \) if local belief monotonicity holds; if it does not hold, \( S(\theta_0) = 0 \).

Now we perform some comparative static analysis. To begin with, we illustrate the sender’s manipulation incentives. In any separating equilibrium, the sender’s action \( a_1 \) signals his type. We let \( \hat{\theta}(a_1) \) express the belief’s dependence on the sender’s action. He solves the problem:

\[
\max_{a_1} \left[ -(a_1 - k_1 \theta - b_1)^2 - \delta(\hat{\theta}(a_1) - k_2 \theta - b_2)^2 \right]
\]

The sender thereby faces a trade-off between optimizing his action \( a_1 \) and manipulating \( \hat{\theta} \). The strength of such manipulation incentives hinges on \( \delta \) and the sensitivity of the belief \( \hat{\theta}(a_1) \). The greater \( \delta \), the stronger the manipulation incentive. It thus entails a larger equilibrium distortion.

**Proposition 3.** In the dominant separating strategy, the equilibrium distortion increases in \( \delta \).

## 6 Applications

In this section, we apply our results to investigate several examples in the literature.

### 6.1 Leading by Example

The classic model of leadership of Hermalin (1998) considers a firm’s leader who wants to incentivize employees to dedicate their efforts in a common activity. Since the firm benefits from all efforts, the leader is motivated to tell employees that all activities deserve their maximum efforts. Consequently, rational employees would disregard the leader’s call. Nevertheless, the leader herself could exert high effort and thereby lead her followers to do the same.

In the model, a team contains \( N \) identical workers, including a leader. Each worker \( n \) puts in effort \( e_n \) for a common endeavor. The value of the common endeavor is \( V = \theta \sum_{n=1}^{N} e_n \), where \( \theta \in \Theta = [0, 1] \) denotes a random productivity factor.
A worker’s utility is \( s_w \times V - \frac{1}{2} e^2 \) where \( s_w \times V \) denotes his wage, which is determined by an exogenous factor \( s_w \) and the common value \( V \), and where \( \frac{1}{2} e^2 \) is the disutility from exerting effort. The leader’s utility is \( s_l \times V - \frac{1}{2} e^2 \) where \( s_l + (N-1)s_w = 1 \). The leader privately observes \( \theta \). Next, the leader expends effort in front of the other workers who make inferences concerning \( \theta \) based on the leader’s effort. Let \( e(\theta) \) denote the leader’s strategy in equilibrium, and \( \hat{\theta} \) denote the followers’ belief. Each worker \( n \) solves the following problem

\[
\max_{e_n} \left[ s_w \hat{\theta}(e_n + \sum_{j \neq n} e_j) - \frac{e_n^2}{2} \right]
\]

by choosing \( e_n = s_w \hat{\theta} \). Given this, the leader’s payoff from taking effort \( e \) when the true state is \( \theta \) and the followers infer \( \hat{\theta} \) is given by:

\[
V(\theta, \hat{\theta}, e) = s_l \theta (e + (N-1)s_w \hat{\theta}) - \frac{e^2}{2}
\]

which satisfies all the regularity conditions in Mailath (1987). In particular, it satisfies belief monotonicity

\[
V_2(\theta, \hat{\theta}, e) = (N-1)s_w s_l \theta > 0.
\]

The leader solves

\[
\max_{e(\hat{\theta})} \left[ s_l \theta (e(\hat{\theta}) + (N-1)s_w \hat{\theta}) - \frac{e^2(\hat{\theta})}{2} \right]
\]

with the first order condition

\[
e'(\theta) = \frac{s_l (1 - s_l) \theta}{e - s_l \theta}
\]

satisfying the general differential equation (2). Matching coefficients, we have \( r = s_l (1 - s_l) > 0 \), \( \theta_0 = 0 \in \Theta \). Therefore, by theorem 2 and 3, the unique dominant PBE is linear, which coincides with Lemma 3 and Proposition 5 of Hermalin (1998).\(^{11}\) Yet this result hinges critically on the assumption \( \Theta = [0, 1] \). If the productivity factor is bounded away from 0 (e.g. \( \Theta = [1, 2] \)), the dominant equilibrium is no longer linear. This case was not analyzed by Hermalin (1998).

\(^{11}\)Zhou (2016) applies Hermalin (1998)’s model to explore leadership within hierarchical organizations. His analysis heavily depends on the linear equilibrium.
To deepen our understanding, we rewrite the leader’s problem as

$$V = -\frac{1}{2}(e - sl\theta)^2 + \frac{1}{2}sl^2\theta^2 + r\theta \hat{\theta}. $$

If the leader is myopic and ignores the inferential impact of his effort, he optimally chooses $e = sl\theta$, i.e., his bliss point. For a strategic leader, he can benefit from inducing a higher belief $\hat{\theta}$. The marginal benefit of inducing a higher belief is $\frac{\partial V}{\partial \hat{\theta}} = r\theta$ where $r$ is the marginal impact. Thus, the higher the productivity factor $\theta$, the stronger the incentive. As he desires to induce a higher belief, $e = sl\theta$ cannot be sustained in equilibrium and his effort is biased upward. But as his effort diverges from his bliss point $e = sl\theta$, it incurs a cost in $-\frac{1}{2}(e - sl\theta)^2$. In equilibrium, the marginal cost of exerting more effort must equal the marginal benefit of inducing a higher belief, for all separating strategies. Among them, the dominant strategy is the one closest to the leader’s bliss point $e = sl\theta$.

Suppose $\Theta = [1, 2]$. The dominant strategy is between the leader’s bliss point and the linear strategy (Figure 3) and thereby features a uniformly lower effort than the linear strategy. In the linear strategy, the marginal cost of inducing a higher belief

$$\left. \frac{\partial}{\partial \hat{\theta}} \left[ -\frac{1}{2}(e(\hat{\theta}) - sl\theta)^2 \right] \right|_{\hat{\theta} = \theta}$$

is a first-order effect as $e(\theta) - sl\theta > 0$ for all $\theta \in \Theta$. The linear slope is pinned down such that this first-order effect balances the marginal benefit $r\theta$. In contrast, the dominant strategy features $e(1) = sl$. If $e(\theta)$ were to increase linearly, the marginal

---

12See the close form in the proof of Theorem 2 in the appendix.
cost of inducing a higher belief

\[
\frac{\partial}{\partial \hat{\theta}} \left[ -\frac{1}{2} (e(\hat{\theta}) - s_l \theta)^2 \right] \bigg|_{\hat{\theta} = \theta}
\]

would be zero at \( \theta = 1 \) (a second-order effect), which falls short of the non-zero marginal benefit. Thus, as \( \theta \) approaches 1, the slope of \( e(\theta) \) tends to infinity. As \( \theta \) increases, \( e(\theta) \) diverges away from \( s_l \theta \). As the equilibrium distortion \( e(\theta) - s_l \theta \) grows larger, it requires less increment in \( e(\theta) - s_l \theta \) to balance the marginal benefit. But as the equilibrium distortion in the dominant strategy is smaller than that of the linear strategy, the slope \( e'(\theta) \) of the dominant strategy is larger than that of the linear strategy. Therefore, the dominant strategy converges to the linear one as \( \theta \) gets large. Consequently, the dominant strategy is non-linear.

What is different when \( \Theta = [0, 1] \)? If \( \Theta = [0, 1] \), at \( \theta = 0 \), the marginal benefit of inducing a higher belief, \( r\theta \), is also zero, which allows the linear strategy with \( e(0) = 0 \) to grow linearly at \( \theta = 0 \). As the linear strategy satisfies the initial condition \( e(0) = 0 \), the linear strategy coincides with the dominant strategy.

### 6.2 Data Linkages

Argenziano and Bonatti (2021) consider a dynamic model of behavior-based price discrimination. A consumer sequentially interacts with two firms. In each period \( t \in \{1, 2\} \), the active firm sets a price \( p_t \) and a quality level \( y_t \) and the consumer chooses the quantity \( q_t \) to consume. A data linkage allows the second firm to observe the first-period interaction outcome \((p_1, y_1, q_1)\). With the information acquired, the second firm tailors its quality level and price to the consumer’s type.

In each period, the consumer’s utility is given by

\[
U(p_t, y_t, q_t) = (\theta + b_t y_t - p_t)q_t - q_t^2
\]

\( \theta \in \Theta = [m, M] \) denotes the consumer’s type, i.e., his baseline consumption level before adjusting for price and quality. \( b_t \in [0, \sqrt{2}] \) is common knowledge and represents the sensitivity of the consumer’s valuation to the quality of firm \( t \)'s product. \( b_t y_t - p_t \) is called the terms of trade that firm \( t \) offers to the consumer. Firm \( t \)'s profits are

\[
\Pi(p_t, y_t, q_t) = p_t q_t - \frac{q_t^2}{2}
\]
Dominant strategy
\[ q_1 = \theta + b_1 y_1 - p_1 \]

Figure 4: Consumer’s Strategy for \( \lambda_2 < 0 \) (Left), \( \lambda_2 > 0 \) and \( 0 \notin \Theta \) (Right)

To maximize profits, the second firm sets its terms of trade to be \( \hat{\lambda}_2 \theta \) (we skip the calculations), where \( \hat{\theta} \) is the inferred type and \( \lambda_2 = \frac{b_2 - 1}{2 - b_2^2} \in \left[ -\frac{1}{2}, \infty \right) \). Then, the consumer of type \( \theta \) gets payoff

\[
V(\theta, \hat{\theta}, q_1) = (\theta + b_1 y_1 - p_1)q_1 - \frac{q_1^2}{2} + \frac{1}{2}(\theta + \lambda_2 \hat{\theta})^2.
\]

Notice that the belief monotonicity condition might be violated when \( \lambda_2 \leq 0 \).

The consumer hence solves the problem

\[
\max_{q_1(\hat{\theta})} \left[ (\theta + b_1 y_1 - p_1)q_1(\hat{\theta}) - \frac{q_1^2(\hat{\theta})}{2} + \frac{1}{2}(\theta + \lambda_2 \hat{\theta})^2 \right].
\]

Matching the coefficients to our model (1), we have \( r = \lambda_2(1 + \lambda_2) \), \( \theta_0 = 0 \). Let \( S = q_1 - (\theta + b_1 y_1 - p_1) \). There are two linear IC strategies \( S = \lambda_2 \theta \) and \( S = -(\lambda_2 + 1) \theta \).

By Theorem 2 and 3, in the dominant PBE, the consumer’s strategy is linear iff \( \lambda_2 > 0 \) and \( 0 \notin \Theta \). When the consumer’s dominant strategy is indeed linear, it coincides with the linear equilibrium of Proposition 2 in Argenziano and Bonatti (2021).

Yet, our Theorem 2 implies that the dominant strategy can be non-linear.\(^{13}\) We

\[^{13}\text{When } \lambda_2 < 0, \text{ the dominant strategy is}
\]

\[
\frac{(S - \lambda_2 \theta)^\lambda_2}{(S + (\lambda_2 + 1) \theta)^{-(\lambda_2 + 1)}} = \frac{(-\lambda_2 M)^\lambda_2}{(\lambda_2 M + M)^{-(\lambda_2 + 1)}} \quad (S \leq 0).
\]

When \( \lambda_2 > 0 \) and \( 0 \notin \Theta \), the dominant strategy is

\[
\frac{(\lambda_2 \theta - S)^\lambda_2}{(S + (1 + \lambda_2) \theta)^{-(1 + \lambda_2)}} = \frac{(\lambda_2 m)^\lambda_2}{[(1 + \lambda_2) m]^{-(1 + \lambda_2)}} \quad (S \geq 0).
\]

We need to additionally check whether the strategy in Theorem 2 satisfies problem (4)’s second order condition. It turns out to be true for both solutions above.
plot the remaining cases in Figure 4. In the first period, the consumer strategically consumes less than his ideal quantity $\theta + b_1 y_1 - p_1$ if $\lambda_2 < 0$ but consume more than his ideal quantity if $\lambda_2 > 0$. When $\lambda_2 < 0$, the second firm’s terms of trade $\lambda_2 \hat{\theta}$ is decreasing in belief. As the consumer’s second-period utility is $\frac{1}{2}(\theta + \lambda_2 \hat{\theta})^2$, the consumer bears a cost of inducing a higher belief. In equilibrium, as the consumer consumes less than the ideal quantity, the marginal benefit of consuming more in the first period must balance the marginal cost of inducing a higher belief in the second period. In contrast, when $\lambda_2 > 0$, the consumer benefits from inducing a higher belief. In equilibrium, as the consumer consumes more than the ideal quantity, the marginal cost of consuming more in the first period must balance the marginal benefit of inducing a higher belief in the second period.

### 6.3 Strategic Communication with Lying costs

Kartik (2009) and Kartik et al. (2007) analyze a model of strategic communication between an informed but upwardly biased sender (he) and an uninformed receiver (she). The sender bears a cost of misreporting or lying about his private information. The cost may stem from moral constraints, legal penalties, or fabrication costs. In this setting, inflated language arises, where we say the language is inflated if the sender’s message is biased above the state. To preserve the flavor of inflated language in equilibrium without getting into technical details, we assume that $\Theta = [0, 1]$ as in Kartik (2009).

In the model, a sender is privately informed about the state $\theta \in \Theta$. After observing the state, he sends a message $m$ to a receiver who then takes an action $a_2$. The sender’s payoff is

$$U^S = -k(m - \theta)^2 - (a_2 - a_2^S(\theta))^2$$

where $a_2^S(\theta) = \lambda \theta + b$ is his ideal action for the receiver and $k(m - \theta)^2$ denotes the lying cost with true state $\theta$ and message $m$. The receiver’s payoff is maximized when her action $a_2$ matches the state, i.e., $a^R(\theta) = \theta$. Given the receiver’s best response, the sender gets payoff

$$V(\theta, \hat{\theta}, m) = -k(m - \theta)^2 - [\hat{\theta} - (\lambda \theta + b)]^2,$$

when the belief is $\hat{\theta}$. Note that both local belief monotonicity and the sign condition in Kartik et al. (2007) might fail. Plugging in the coefficients in our model (1), we have $r = \frac{\lambda - 1}{k}$ and $\theta_0 = -\frac{b}{\lambda - 1}$. 

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\[ a_2^S = \lambda \theta + b \]

Figure 5: Sender’s Strategy when \( \theta_0 \in \Theta \)
\( \lambda > 1 \) (Left) and \( \lambda < 1 \) (Right)

Kartik et al. (2007) consider the case where \( \lambda = 1 \) and \( b > 0 \). By Theorem 2 and 3, the sender’s strategy is non-linear

\[ m + \frac{b}{k} \ln\left(\frac{b}{k} - m + \theta\right) = \frac{b}{k} \ln\left(\frac{b}{k}\right), \]

as in solution (6) in Kartik et al. (2007). The language in equilibrium is inflated as \( m \geq \theta \).

Yet, this feature of inflated language relies critically on local belief monotonicity. That is, the language is inflated since \( a_2^S(\theta) > a_2^R(\theta) \) for all \( \theta \in \Theta \). What happens if local belief monotonicity fails? (Then, the sender’s preference \( a_2^S(\theta) \) is biased upward for some states but biased downward for other states.) Can we generalize this result? As we analytically solve the dominant strategy in the proof of Theorem 2, we can directly examine the direction of equilibrium distortion. It turns out that both inflated and deflated language can appear and the direction of language distortion depends only on how the sender’s bliss point \( a_2^S(\theta) \) is biased from the receiver’s ideal point \( a_2^R(\theta) \). In particular, the language is inflated (deflated) whenever the sender’s bliss point is above (below) the state.

**Proposition 4.** In the dominant PBE, the sender’s language is inflated if and only if \( a_2^S(\theta) > a_2^R(\theta) \).

This result is not limited to the model of lying cost. A similar conclusion holds for all quadratic signaling games. By Theorems 2 and 3, the dominant PBE is linear iff \( \lambda > 1 \) and \( \theta_0 \in \Theta \). In Figure 5, we plot the case for \( \lambda > 1 \) when the dominant strategy is linear and the case \( \lambda < 1 \) when the dominant strategy is non-linear. We can see
that the direction of language distortion is indeed fully aligned with the direction of preference bias $a_2^S(\theta) - a^R(\theta)$.

### 6.4 Transferable Control and Learning by Delegation

The classic delegation problem considers a principal (she) who faces an informed but biased agent (he). The principal delegates control to the agent to utilize his local knowledge and is unable to commit to contingent transfers. In many real-life examples, the principal cannot contractually commit not to withdraw control in the future (Aghion et al., 2004). After learning the agent’s local knowledge from the delegated decision, the principal can reclaim control and make decisions by herself. Aghion et al. (2004) analyze the case for a discrete state. In this section, we investigate how transferable control and learning by delegation affect the delegated decision for a continuous state.

In a two-period delegation model, the principal delegates authority in the first period and retakes control in the second one. $\theta$ denotes the state of the world, $\theta \in \Theta = [0, 1]$. In the first period, the agent privately observes the state $\theta$ and makes a decision $a_1 \in \mathbb{R}$. In the second period, the principal makes a decision $a_2 \in \mathbb{R}$ based on the agent’s decision $a_1$. The principal’s and the agent’s payoffs depend on the implemented decisions and the state. The principal wants her decision $a_2$ to match the state, while the agent’s ideal point is $a_1^S = a_2^S = \theta + b$ where $b > 0$ measures the preference bias. The agent’s payoff is $U^S = -(a_1 - \theta - b)^2 - \delta (a_2 - \theta - b)^2$.

Given the principal’s belief $\hat{\theta}$, the agent’s payoff is

$$V(\theta, \hat{\theta}, a_1) = -(a_1 - \theta - b)^2 - \delta (\hat{\theta} - \theta - b)^2.$$
It does not satisfy belief monotonicity whenever $|b| < 1$. Matching the coefficients to our model (1), we have $r = 0$. By Theorem 2, the dominant strategy is non-linear (see Figure 6),

$$\theta + S + \delta b \ln(\delta b - S) = \delta b \ln(\delta b).$$

(5)

The symmetry of $\delta$ and $b$ implies they have the same effect on the distortion $S$.\footnote{It is attempting to disregard the effect of $b$ on $S$ as trivial since $b$ looks like a mere scale factor. Yet it is not the case. After descaling, $x \triangleq \frac{x}{\delta b}$ is decreasing in $\delta$ and $b$.} We immediately obtain a corollary of Proposition 3.

**Corollary 1.** *In the dominant PBE, the equilibrium distortion increases in both $\delta$ and $b$.*

As $\delta$ tends to zero, the optimal equilibrium degenerates to the standard static delegation

$$a_1 = \theta + b$$

as in Dessein (2002).

Up to now, we have focused on the dominant PBE that maximizes the sender/agent’s payoff at every state. In this example, we explore how the receiver/principal ranks different separating PBE. Does the dominant PBE yield the highest possible expected payoff to the principal? To address this question, it is necessary to impose some assumption on the state distribution and on the principal’s preference about $a_1$. Following the classic example in Crawford and Sobel (1982), we assume the state is uniformly distributed and $U^R = -(a_1 - \theta)^2 + g(\theta, a_2)$ where $g(\theta, a_2)$ is maximized at $a_2 = \theta$ for all $\theta$. In this setting, we can show that the (agent) dominant PBE is optimal for the principal.

**Proposition 5.** *The (agent) dominant PBE is also optimal for the principal.*

By the proof of Theorem 2, we can analytically solve all IC strategies. Some of them are discontinuous. To prove Proposition 5, we first show that all discontinuous strategies are Pareto-dominated by the linear strategy. (The linear strategy is increasing.) Second, we show all decreasing continuous strategies cannot be admissible for any PBE. Third, among all increasing continuous strategies, we show the (agent) dominant strategy is uniformly closest to the principal’s ideal point.
7 Conclusion

This paper studies dominant separating equilibria in quadratic signaling games. One assumption we impose is a quadratic form of the preference. This form succinctly captures some key properties of concave preferences and allows for closed form solutions. It is thereby widely used in the literature. Although Mailath (1987) allows for more general preferences, he requires belief monotonicity which precludes various applications. We relax belief monotonicity to fill this gap. Nevertheless, a natural direction for future research is to generalize our conclusions beyond quadratic games.

Another interesting question for future research is how to find the dominant partial separating equilibria. As shown by Kartik (2009), a fully separating equilibrium may not always exist and sometimes we need to focus on partially separating equilibria. It is likewise interesting to rank different partial separating equilibria.
References


A Omitted Proofs

**Proof of Lemma 1.** The proof that $a_1$ must be differentiable is contained in lemma 1 of Kartik (2009). Given the differentiability, (2) is the first order condition of

$$-[a_1(\theta) - b_1]^2 - \delta(\theta - b_2)^2 \leq -[a_1(\theta) - k_1\theta - b_1]^2 - \delta(\theta - k_2\theta - b_2)^2$$

for $\forall \theta$.

**Proof of Lemma 2.** If the sender chooses $a_1 < \min_{\theta \in \Theta} a_1(\theta)$ at some state $\theta$, the principal believes $\theta = \hat{\theta}$. Hence, the sender’s payoff is

$$U^S = -(a_1 - k_1\theta - b_1)^2 - \delta(\theta - k_2\theta - b_2)^2 < -\left(\min_{\theta \in \Theta} a_1(\theta) - k_1\theta - b_1\right)^2 - \delta(\theta - k_2\theta - b_2)^2$$

by $B \in A$. That is, this action is strictly worse than mimicking state $\theta = \hat{\theta}$. Therefore, this deviation is not profitable for the sender.

On the other hand, if the sender chooses $a_1(\theta) > \max_{\theta \in \Theta} a_1(\theta)$ at some state $\theta$, the principal believes $\theta = \bar{\theta}$. The sender’s payoff is

$$U^S = -(a_1 - k_1\theta - b_1)^2 - \delta(\bar{\theta} - k_2\theta - b_2)^2 < -\left(\max_{\theta \in \Theta} a_1(\theta) - k_1\theta - b_1\right)^2 - \delta(\bar{\theta} - k_2\theta - b_2)^2$$

by $B \in A$. This action is strictly worse than mimicking state $\theta = \bar{\theta}$.

For the next two lemmas, we state the case for $k_1 > 0$. The other case is symmetric around the $\theta$-axis.

**Lemma 3. Aligned Monotonicity 1.** For any $\theta \in \Theta$, if there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$, there exists $\theta' \in (\theta - k_1\epsilon, \theta)$ such that $S(\theta') > 0$, then we must have $S(\theta) \geq 0$ and it satisfies $|S(\theta)|\left|\frac{dS(\theta)}{d\theta}\right| + k_1 = r\theta + \delta b_2$.

**Lemma 4. Aligned Monotonicity 2.** For any $\theta \in \Theta$, if there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$, there exists $\theta' \in (\theta, \theta + k_1\epsilon)$ such that $S(\theta') < 0$, then we must have $S(\theta) \leq 0$ and it satisfies $-|S(\theta)|\left|\frac{dS(\theta)}{d\theta}\right| + k_1 = r\theta + \delta b_2$.

**Proof of Lemma 3, 4.** Since the proof of these two lemmas are similar, we just prove Lemma 3. Let $\mathcal{I} = (\theta - k_1\epsilon, \theta)$. Towards a contradiction, suppose $S(\theta) < 0$. Then at state $\theta - k_1\epsilon$ with $S(\theta - k_1\epsilon) > 0$, if the sender mimics state $\theta$ by choosing $-|S(\theta)|$, he gets:

$$-(|S(\theta)| - k_1^2\epsilon)^2 - \delta(\theta - k_2(\theta - k_1\epsilon) - b_2)^2.$$  \hspace{1cm} (6)
If the sender chooses $S(\theta - k_1 \epsilon)$, he gets:

$$- S^2(\theta - k_1 \epsilon) - \delta(\theta - k_1 \epsilon - k_2(\theta - k_1 \epsilon) - b_2)^2. \quad (7)$$

By assumption, the set $\{\theta' \in \mathcal{I} : S(\theta') > 0\}$ is a dense set in $\mathcal{I}$. So we have

$$|S(\theta)| \left[ \frac{d|S(\theta)|}{d\theta} + k_1 \right] = \delta(k_2 - 1) \theta + \delta b_2.$$

Therefore, the difference between equation (6) and equation (7) is

$$4k_1^2 |S(\theta)| \epsilon - \mathcal{O}(\epsilon^2) > 0$$

when $\epsilon$ is sufficiently small. This implies that if $S(\theta) < 0$, sender at state sufficiently close to $\theta$ has incentive to mimic $\theta$ and hence the incentive compatibility constraint is violated. So we must have $S(\theta) > 0$ and satisfies differential equation $|S(\theta)| \left[ \frac{d|S(\theta)|}{d\theta} + k_1 \right] = \delta(k_2 - 1) \theta + \delta b_2.$

**Proof of Proposition 1.** Lemma 3 and 4 put two more restrictions on the discontinuity of any IC strategy. First, any IC strategy contains at most one discontinuity. Second, wherever an IC strategy is discontinuous, the direction of its jump must be aligned with the sender’s preference. That is, if $k_1 > 0$ ($k_1 < 0$), it must jump upwards (downwards).

**Lemma 5. Contraction Transform.** Let $\Theta = \Theta_1 \cup \Theta_2$, $\Theta_1 \cap \Theta_2 = \emptyset$. Suppose $S_1(\theta)$ and $S_2(\theta)$ are IC strategies defined over $\Theta$, $\Theta_2$, respectively. If

1. for $\forall \theta \in \Theta_2$, $|S_2(\theta)| \leq |S_1(\theta)|$,

2. for $\forall \theta_1 \in \Theta_1$, $\forall \theta_2 \in \Theta_2$, $|S_1(\theta_2) + k_1(\theta_2 - \theta_1)| \leq |S_2(\theta_2) + k_1(\theta_2 - \theta_1)|$,

then

$$S(\theta) \triangleq \begin{cases} S_1(\theta), & \text{if } \theta \in \Theta_1 \\ S_2(\theta), & \text{if } \theta \in \Theta_2 \end{cases}$$

is incentive compatible.
In other words, if an IC strategy $S_1$ contracts over $\Theta_2$ towards the bliss point of $\Theta_2$ while moving away from the bliss point of $\Theta_1$ (see Figure 7), it is still incentive compatible.

**Proof of Lemma 5.** Let $U^S(\theta'; \theta)$ denote the payoff of sender $\theta$ if he mimics state $\theta'$. Let $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$. By design, sender with $\forall \theta_1$ ($\theta_2$) does not mimic any state in $\Theta_1$ ($\Theta_2$). The sender with state $\theta_2$ does not mimic $\theta_1$ because

$$U^S(\theta_1; \theta_2) = -(S(\theta_1) + k_1(\theta_1 - \theta_2))^2 - \delta(\theta_1 - k_2\theta_2 - b_2)^2$$

$$\leq -(S(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2$$

$$\leq -(S_2(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2$$

$$= -(S(\theta_2))^2 - \delta(\theta_2 - k_2\theta_2 - b_2)^2$$

$$= U^S(\theta_2; \theta_2)$$

where the first inequality is by $S_1$ being IC.

The sender with state $\theta_1$ does not mimic $\theta_2$ because

$$U^S(\theta_2; \theta_1) = -(S(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2$$

$$= -(S_2(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2$$

$$\leq -(S_1(\theta_2) + k_1(\theta_2 - \theta_1))^2 - \delta(\theta_2 - k_2\theta_1 - b_2)^2$$

$$\leq -(S_1(\theta_1))^2 - \delta(\theta_1 - k_2\theta_1 - b_2)^2$$

$$= U^S(\theta_1; \theta_1)$$

where the second inequality is by $S_1$ being IC.
For the following lemma, WLOG, we only analyze the case with positive $S$, $q$, $n$ and $p$. Other cases can be addressed similarly.

**Lemma 6. Monotone Expansion.** Suppose $q$, $n$, $p$, $q\theta^0 + n \geq 0$. Let $S(\theta; q, n, p)$ denote the solution for

$$\frac{dS}{d\theta} = \frac{q\theta + n}{S} + p, \quad S \geq 0$$

with initial value $S(\theta^0; q, n, p) = 0$. $S(\theta; q, n, p)$ monotonically increases with all parameters ($q$, $n$ and $p$) for any $\theta \geq \theta^0$.

**Proof of Lemma 6.** WLOG, we only prove the monotonicity WRT $q$. Let $0 \leq q_1 \leq q_2$. We shall show $S(\theta; q_1, n, p) \leq S(\theta; q_2, n, p)$ for $\forall \theta \geq \theta^0$.

By way of contradiction, suppose $\exists \theta^* > \theta^0$ s.t. $S(\theta^*; q_1, n, p) > S(\theta^*; q_2, n, p)$. By $S(\theta^0; q_1, n, p) = S(\theta^0; q_2, n, p) = 0$ and

$$S(\theta; q, n, p) = \int_{\theta^0}^{\theta} \frac{\partial S(\theta; q, n, p)}{\partial \theta} d\theta,$$

there must exist $\tilde{\theta} \in (\theta^0, \theta^*)$ s.t.

$$S(\tilde{\theta}; q_1, n, p) > S(\tilde{\theta}; q_2, n, p)$$

and

$$\frac{\partial S(\theta; q_1, n, p)}{\partial \theta} \Big|_{\theta = \tilde{\theta}} > \frac{\partial S(\theta; q_2, n, p)}{\partial \theta} \Big|_{\theta = \tilde{\theta}}$$

contradicting

$$\frac{\partial S(\theta; q_1, n, p)}{\partial \theta} \Big|_{\theta = \tilde{\theta}} = \frac{q_1 \theta + n}{S(\tilde{\theta}; q_1, n, p)} + p$$

$$< \frac{q_1 \theta + n}{S(\tilde{\theta}; q_2, n, p)} + p$$

$$\leq \frac{q_2 \theta + n}{S(\tilde{\theta}; q_2, n, p)} + p$$

$$= \frac{\partial S(\theta; q_2, n, p)}{\partial \theta} \Big|_{\theta = \tilde{\theta}}$$

\[\square\]

**Proof of Theorem 2, 3.** In this proof, we only consider continuous IC strategy and we shall prove discontinuous ones are sub-optimal in the Online Appendix. By (3), we only need to consider the case $k_1 > 0$, since cases for $k_1 > 0$ and $k_1 < 0$ are symmetric
around $\theta$-axis. The SOC requires
\[-2\left(\frac{dS(\theta)}{d\theta} + k_1\right)^2 - 2S\frac{d^2S(\theta)}{d\theta^2} - 2\delta \leq 0 \tag{8}\]

We shall solve the differential equation case by case.

**Case 1: $r = 0$**

(3) is
\[SdS = (\delta b_2 - k_1 S)d\theta\]

The solution of this differential equation is:
\[k_1^2\theta + k_1 S + \delta b_2 \ln |\delta b_2 - k_1 S| = C\]

and a special solution $k_1 S = \delta b_2$. The SOC is $\frac{k_1 b_2}{S} + 1 \geq 0$. We only analyze the case for $b_2 > 0$\(^{15}\), since the cases for $b_2 < 0$ can be addressed centro-symmetrically. When

\[k_1 > 0 \text{ and } b_2 > 0, \text{ SOC requires } S \geq 0 \text{ or } S \leq -k_1 b_2. \text{ There is one strategy crossing } (\theta, S) = (m, 0)\]

\[k_1^2(\theta - m) + k_1 S + \delta b_2 \ln(\delta b_2 - k_1 S) = \delta b_2 \ln(\delta b_2) \quad (S \geq 0). \tag{9}\]

It is uniformly closest to sender’s bliss point $S = 0$ among all IC strategy $S \geq 0$. Next we show it dominates any IC strategy $S < 0$. For any solution $k_1^2\theta + k_1 S + \delta b_2 \ln(\delta b_2 - k_1 S) = C$, let $S^+(\theta; C)$, $S^-(\theta; C)$ denote the positive branch ($S \geq 0$) and the negative

\(^{15}\)When $b_2 = 0$, the optimal solution is trivially $S(\theta) = 0$.\]
one \((S \leq 0)\). By monotone expansion Lemma 6, \(|S^+(\theta; C)| \leq |S^-(\theta; C)|\). Thus, solution \((9)\) dominates all IC strategies \(S \leq 0\) defined over \(\Theta\). Thus, the dominant strategy is \((9)\), which is continuous and unique. By \(\frac{da_1}{d\theta} = \frac{da_2}{S} > 0\), it is monotonic. Moreover, by \(k_1 > 0\)
\[
A^S_1 \subset A_1.
\]

Thus, it supports a PBE by Lemma 2.

**Preliminary analysis for** \(r \neq 0\)

Define \(\phi \triangleq \theta - \theta_0\). DE (3) is
\[
S(\phi)[\frac{dS(\phi)}{d\phi} + k_1] = r\phi. \tag{10}
\]
which is centrosymmetric around \((\phi, S) = (0, 0)\). By
\[
\frac{d^2 S(\phi)}{d\phi^2} = \frac{d(\frac{dS}{d\phi})}{dS} \frac{dS}{d\phi} = r\left(\frac{1}{S} - \frac{\phi}{S^2} \frac{dS}{d\phi}\right),
\]
SOC (8) is reduced to
\[
\delta[k_2 + k_1(k_2 - 1)\frac{\phi}{S}] \geq 0 \tag{11}
\]
To fully solve DE (10), let \(w \triangleq \frac{S}{\phi}\). DE (10) can be reduced to
\[
\frac{wdw}{w^2 + k_1w - r} = -\frac{d\phi}{\phi} \tag{12}
\]
The form of general solutions depends on \(\Delta\). Linear strategy exists if and only if \(\Delta \geq 0\). If \(\Delta > 0\), there exist two linear special solutions \(S = w_1\phi\) and \(S = w_2\phi\), where \(w_1\) and \(w_2\) denote solutions for the equation \(w^2 + k_1w - r = 0\). WLOG, let us assume \(|w_1| \leq |w_2|\).
\[
w_1 + w_2 = -k_1 \tag{13}
\]
\[
w_1 \times w_2 = -r \tag{14}
\]
Moreover, \(S = w_1\phi\) and \(S = w_2\phi\) satisfy the SOC (8) trivially: \(-2\left(\frac{da_1(\theta)}{d\theta}\right)^2 - 2\delta \leq 0\). We further rewrite DE (12)
\[
\frac{wdw}{(w - w_1)(w - w_2)} = -\frac{d\phi}{\phi}
\]
\[
\frac{w_1}{w_1 - w_2} \frac{dw}{w - w_1} - \frac{w_2}{w_1 - w_2} \frac{dw}{w - w_2} = -\frac{d\phi}{\phi}
\]
29
Figure 9: Solutions for $r > 0$.

The general solutions take the form

$$\frac{|S - w_1\phi|^{w_1}}{|S - w_2\phi|^{w_2}} = C$$

If $\Delta = 0$, there is a unique linear special solutions $S = w_1\phi$, where $w_1 = w_2 = -\frac{k_1}{2}$ is the solution for equation $w^2 + k_1w - r = 0$. This linear solution also satisfies the SOC. We further rewrite DE (12)

$$\frac{wdw}{(w - w_1)^2} = -\frac{d\phi}{\phi}$$

$$\frac{dw}{w - w_1} + \frac{w_1 dw}{(w - w_1)^2} = -\frac{d\phi}{\phi}$$

The general solutions take the form

$$\ln |S - w_1\phi| - \frac{w_1\phi}{S - w_1\phi} = C$$

**Case 2: $r > 0$**

In this case, $\Delta > 0$. In figure 9, we plot the solution for $r > 0$.

Suppose $\theta_0 \notin \Theta$. WLOG, we only consider $\theta_0 < m$. There is one solution crossing $(\theta, S) = (m, 0)$ among all solutions (15):

$$\frac{(w_1\phi - S)^{w_1}}{(S - w_2\phi)^{w_2}} = \frac{[w_1(m - \theta_0)]^{w_1}}{[w_2(\theta_0 - m)]^{w_2}} \quad (S \geq 0, \phi \in [m - \theta_0, M - \theta_0]).$$

$$\frac{(w_1\phi - S)^{w_1}}{(S - w_2\phi)^{w_2}} = \frac{[w_1(m - \theta_0)]^{w_1}}{[w_2(\theta_0 - m)]^{w_2}} \quad (S \geq 0, \phi \in [m - \theta_0, M - \theta_0]).$$
Figure 10: Solutions for $r < 0$, $\Delta > 0$.

It satisfies the SOC. It is uniformly closest to sender’s bliss point $S = 0$ among all IC strategies $S \geq 0$. By monotone expansion Lemma 6, it dominates any IC strategy with $S < 0$. Thus, the dominant strategy is (17), which is continuous, non-linear and unique. By $\frac{\partial w_1}{\partial \theta} = \frac{r \phi}{S} > 0$, it is monotonic. Moreover, it supports a PBE by Lemma 2.

Suppose $\theta_0 \in \Theta$. Branches $\frac{(w_1 \phi - S)^{w_1}}{(S-w_2 \phi)^{w_2}} = C$, $\frac{(S-w_1 \phi)^{w_1}}{(w_2 \phi - S)^{w_2}} = C$ cannot be supported on $\Theta$. In addition, Branches $\frac{(S-w_1 \phi)^{w_1}}{(S-w_2 \phi)^{w_2}} = C$, $\frac{(w_1 \phi - S)^{w_1}}{(w_2 \phi - S)^{w_2}} = C$, $S = w_1 \phi$ are uniformly dominated by $S = w_1 \phi$. Thus, the dominant strategy is $S = w_1 \phi$, which is continuous, monotonic, linear and unique. Moreover, by $w_1 > 0$ and $k_1 > 0$,

$$A_1^S \subset A_1.$$ 

Thus, $S = w_1 \phi$ supports a PBE by Lemma 2.

**Case 3: $r < 0$**

We further categorize this case in terms of $\Delta$.

**Case 3.1: $r < 0$, $\Delta > 0$**

In figure 10, we plot the solution for $r < 0$, $\Delta > 0$.

Suppose $\theta_0 \notin \Theta$. If $\theta_0 \geq M$, there is one solution crossing $(\theta, S) = (m, 0)$:

$$\frac{(w_1 \phi - S)^{w_1}}{(w_2 \phi - S)^{w_2}} = \frac{[w_1(m - \theta_0)]^{w_1}}{[w_2(m - \theta_0)]^{w_2}} \quad (S \geq 0, \ \phi \in [m - \theta_0, M - \theta_0]). \quad (18)$$

It satisfies the SOC. It is uniformly closest to sender’s bliss point $S = 0$ among all $S \geq 0$ IC strategies. By monotone expansion property, it dominates any IC strategy with $S < 0$. Thus, the dominant strategy is (18), which is continuous, non-linear and...
unique. By $\frac{d\alpha_1}{d\theta} = \frac{r\phi}{S} > 0$, it is monotonic. Moreover, it supports a PBE by Lemma 2. If $\theta_0 \leq m$, the same argument applies and the optimal one crosses $(\theta, S) = (M, 0)$:

$$\frac{(S - w_1\phi)^{w_1}}{(S - w_2\phi)^{w_2}} = \frac{(-w_1(M - \theta_0))^{w_1}}{(-w_2(M - \theta_0))^{w_2}} \quad (S \leq 0, \, \phi \in [m - \theta_0, M - \theta_0]). \quad (19)$$

Suppose $\theta_0 \in \Theta$. By the argument above, for $\theta \in [m, \theta_0]$, solution (18) is dominant; for $\theta \in [\theta_0, M]$, solution (19) is dominant. Nevertheless, since (18) and (19) are not necessarily on the same integral curve, we need to check whether the combination (18)+(19) is incentive compatible. By the centrosymmetry of DE (10), any integral curve crossing $(\phi, S) = (0, 0)$ is centrosymmetric around $(\phi, S) = (0, 0)$. By DE (10)

$$\frac{d\alpha_1(\theta)}{d\theta} = \frac{dS}{d\phi} + k_1 = \frac{\delta(k_2 - 1)\phi}{S} > 0$$

for the combination (18)+(19). Thus, by Lemma 5, the combination is a contraction transform of the integral curve of (15) where (18) lies (see figure 7) when $\theta_0 > \frac{m + M}{2}$, where (19) lies when $\theta_0 < \frac{m + M}{2}$. Therefore, the combination is IC. Moreover, it is the unique dominant strategy, which is non-linear, monotonic, and continuous. By Lemma 2, it supports a PBE.

The analysis for the remaining cases below is similar.

**Case 3.2:** $r < 0, \Delta = 0$ (See figure 11)

If $\theta_0 > M$, the unique dominant solution crosses $(\theta, S) = (m, 0)$:

---

16They are on the same integral curve iff $\theta_0 = \frac{1}{2}$. 32
Figure 12: Delegation $S(\theta)$

$$\ln(w_1 \phi - S) - \frac{w_1 \phi}{S - w_1 \phi} = \ln(w_1 (m - \theta_0)) + 1 \quad (S \geq 0, \phi \in [m - \theta_0, M - \theta_0]), \quad (20)$$

which satisfies the SOC, is continuous and non-linear. It is monotonic by $\frac{d\alpha_1}{d\theta} = r \frac{\phi}{S} > 0$. By Lemma 2, it supports a PBE. If $\theta_0 < m$, the unique dominant one crosses $(\theta, S) = (M, 0)$:

$$\ln(S - w_1 \phi) - \frac{w_1 \phi}{S - w_1 \phi} = \ln(-w_1 (M - \theta_0)) + 1 \quad (S \leq 0, \phi \in [m - \theta_0, M - \theta_0]), \quad (21)$$

which satisfies the SOC, is continuous and non-linear. It is monotonic by $\frac{d\alpha_1}{d\theta} = r \frac{\phi}{S} > 0$. By Lemma 2, it supports a PBE. If $\theta_0 \in \Theta$, the combination $(20)+(21)$ is the unique dominant strategy, which is non-linear, continuous, monotonic and supports a PBE.

Proof of Proposition 3. For each non-linear optimal IC strategy with $\theta^0 = m$ or $\theta^0 = M$ in the proof of Theorem 2, invoking the monotone expansion lemma could directly yield the result. If the optimal strategy is linear, we can directly check how $|w_1|$ varies with $\delta$ by (14) and (13).

Proof of Proposition 4. The proof follows directly by checking each solution in the proof of Theorem 2 and take $k_1 = 1$ and $b_1 = 0$.

Proof of Proposition 5. By the proof of Theorem 2, any IC strategy belongs to one of the following two cases: (1) $S(\theta)$ is continuous (Figure 12)

$$\theta + S + \delta b \ln |S - \delta b| = C \quad (S \geq 0 \text{ or } S \leq -b) \quad (22)$$
or

\[ S = \delta b; \quad (23) \]

(2) \( S(\theta) \) is discontinuous at some \( \tilde{\theta} \in (0, 1) \) and satisfies:

\[
\begin{align*}
\theta + S + \delta b \ln(\delta b - S) &= C \quad (S \leq -b) \\
\theta + S + \delta b \ln(S - \delta b) &= C' \quad (S > \delta b)
\end{align*}
\]

for \( \theta < \tilde{\theta} \),

\[
\theta + S + \delta b \ln(\delta b - S) = C' \quad (S > \delta b)
\]

for \( \theta > \tilde{\theta} \), where \( C \) and \( C' \) are chosen such that \(-S(\tilde{\theta}^-) = S(\tilde{\theta}^+)\).

By Lemma 7, decreasing strategies

\[
\theta + S + \delta b \ln(\delta b - S) = C \quad S \leq -b \quad (24)
\]

cannot be admissible for any PBE. By Lemma 8, all discontinuous strategies are Pareto-dominated by the linear strategy 23.

Among all increasing continuous strategies, the (agent) dominant strategy is closest to the principal’s ideal point.

\[ \square \]

**Lemma 7.** Strategy (24) is not admissible for a perfect Bayesian equilibrium.

**Proof of Lemma 7.** Since

\[
\frac{dS}{d\theta} = \frac{\delta b}{S} - 1 < -1,
\]

we have

\[ S < -b - \theta. \]

Since \( a_1(\theta) \leq 0 \) on the equilibrium path for all \( \theta \), we will consider a deviation to choose \( a_1 = \theta + b \) at some state \( \theta > 0 \).

If the off-path belief assigns probability 1 to \( \theta = 0 \) when agent deviates to \( a_1 = \theta + b \), agent’s payoff is:

\[
U_{\text{deviation}}^a = -\delta(\theta + b)^2 \geq -(\theta + b)^2 > -S^2 > -S^2 - \delta b^2 = U^a(\theta),
\]

which means agent will deviate to \( a_1 = \theta + b \) no matter what \( \theta \) is.

If the off-path belief assigns probability 1 to \( \theta = 1 \) when agent deviates to \( a_1 = \theta + b \), agent’s payoff is:

\[
U_{\text{deviation}}^a = -\delta(1 - \theta - b)^2.
\]
In addition, for state $\theta \geq \frac{1}{2} - b$, 

$$U^{a}_{\text{deviation}} = -\delta(1 - \theta - b)^2 \geq -\delta(\theta + b)^2 > -S^2 > U^{a}(\theta).$$

Therefore, it is profitable to deviate to $a_1 = \theta + b$ whenever $\theta \geq \frac{1}{2} - b$.

For other possible off-path beliefs, the principal will take a second-period action within the interval of $(0, 1)$. Hence $U^{a}_{\text{deviation}}$ cannot be as low as $\min \{-\delta(\theta + b)^2, -\delta(1 - \theta - b)^2\}$. Therefore, it is always profitable for the agent to deviate to $a_1 = \theta + b$ when $\theta \geq \frac{1}{2} - b$.

**Lemma 8.** Any discontinuous IC strategy admissible for a perfect Bayesian equilibrium is Pareto-dominated by the linear strategy $S = \delta b$.

**Proof of Lemma 8.** Compared with the discontinuous solution, the linear special solution is uniformly closer to the agent’s bliss point. Thus, it suffices to prove the linear special solution dominates from principal’s perspective.

This proof would proceed as follows (see figure 13): 1. Move the left area $S \leq -b$ closer to principal’s bliss point $S = -b$ to form an upper bound to her payoff through linear approximation over interval $[0, \bar{\theta}]$. 2. Move the right area $S > \delta b$ to the left (decrease the constant $C_1$) and then reach an upper bound to principal’s payoff through linear approximation over interval $[\bar{\theta}, \bar{\theta} + D]$ (D is defined as the size of interval when the linear approximation is above $S = \delta b$). Notice that the movement in second
Step relies on the first step since \(|S|\) is continuous at \(\tilde{\theta}\). 3. Assume \(\tilde{\theta} + D \leq 1\). Prove that the upper bound of principal’s payoff by linear approximation is lower than that of special linear solution, \(-(b + \delta b)^2\). 4. Prove that \(\tilde{\theta} + D \leq 1\), otherwise this solution could not be a perfect Bayesian equilibrium.

**Step 1.** Define principal’s payoff over \([0, \tilde{\theta}]\) as \(u_1\) and her payoff over \([0, \tilde{\theta}]\) of linear approximation as \(U_1\). For \(0 \leq \theta \leq \tilde{\theta}, S \leq -b,\)

\[
\frac{dS}{d\theta} = \frac{\delta b}{S} - 1 \geq -(1 + \delta)
\]

\(S(\tilde{\theta}^-) > -b - (1 + \delta)\tilde{\theta}\)

Since \(\frac{d^2S}{d\theta^2} \geq 0\)

\[
\frac{dS}{d\theta} < -\left[1 + \frac{\delta b}{b + (1 + \delta)\tilde{\theta}}\right]
\]

Define \(K\) as

\[
K = 1 + \frac{\delta b}{b + (1 + \delta)\tilde{\theta}}
\]

\(S(\theta) = S(0) + \int_0^\theta \frac{dS(t)}{dt} dt < -b - K\theta\)

\[
u_1 = -\int_0^{\tilde{\theta}} [S(\theta) + b^2] d\theta < -\int_0^{\tilde{\theta}} (K\theta)^2 d\theta = -\frac{1}{3} K^2 \tilde{\theta}^3 = U_1
\]

\(S(\tilde{\theta}^-) = S(0) + \int_0^{\tilde{\theta}} \frac{dS(\theta)}{d\theta} d\theta < -b - K\tilde{\theta}\)

Define \(S_0\) as

\(S_0 = b + K\tilde{\theta}\)

**Step 2.** Define principal’s payoff over \([\tilde{\theta}, 1]\) as \(u_2\) and her payoff over \([\tilde{\theta}, 1]\) of linear approximation as \(U_2\), the solution passing through \((\tilde{\theta}^+, S(\tilde{\theta}^+))\) as \(\theta + S(\theta, C_1) + \delta b \ln[S(\theta, C_1) - \delta b] = C_1\) while the solution passing through \((\tilde{\theta}^+, S_0)\) as \(\theta + S(\theta, C_1^*) + \delta b \ln[S(\theta, C_1^*) - \delta b] = C_1^*\).

For \(\tilde{\theta} \leq \theta \leq 1, S > \delta b,\)

\[
\frac{\partial S(\theta, C)}{\partial C} = -\frac{\partial S(\theta, C)}{\partial \theta} > 0
\]

Since \(S(\tilde{\theta}^+) = -S(\tilde{\theta}^-) > S_0,\)

\(C_1 > C_1^*\)
Since $\frac{\partial^2 S(\theta, C_1^*)}{\partial \theta^2} > 0$

\[S(\theta, C_1^*) > S(\theta, C_1^*) \geq S_0 + (\theta - \tilde{\theta})\frac{\partial S(t, C_1^*)}{\partial t} \bigg|_{t=\tilde{\theta}^+} = S_0 + (\theta - \tilde{\theta})(\frac{\delta b}{S_0} - 1)\]

Define $k$ as

\[k = -\frac{\partial S(t, C_1^*)}{\partial t} \bigg|_{t=\tilde{\theta}^+} = 1 - \frac{\delta b}{S_0}\]

Define $D$ as size of interval when the linear approximation is above $S = \delta b$

\[S_0 + D(\frac{\delta b}{S_0} - 1) = \delta b\]

\[D = S_0\]

Therefore, for $\theta \in [\tilde{\theta}, \tilde{\theta} + D]$

\[S(\theta, C_1) > S(\theta, C_1^*) \geq S_0 - k(\theta - \tilde{\theta}) \geq \delta b\]

while $S(\theta, C_1) > \delta b$ for $\theta \in [\tilde{\theta} + D, 1]$. Here we assume $\tilde{\theta} + D \leq 1$ and we will prove that this assumption is guaranteed in step 3.

\[u_2 = -\int_{\tilde{\theta}}^{1} [S(\theta, C_1) + b]^2 d\theta \]

\[< -\int_{\tilde{\theta}}^{1} [S(\theta, C_1^*) + b]^2 d\theta \]

\[< -\int_{\tilde{\theta}}^{\tilde{\theta} + D} [S_0 - k(\theta - \tilde{\theta}) + b]^2 d\theta - \int_{\tilde{\theta} + D}^{1} (\delta b + b)^2 d\theta \]

\[= -\int_{0}^{D} [kt + (b + \delta b)]^2 dt - (b + \delta b)^2(1 - \tilde{\theta} - D) \]

\[= U_2\]

**Step 3.** To prove $u_1 + u_2 < -(b + \delta b)^2$, it suffices to prove

\[U_1 + U_2 < -(b + \delta b)^2 \quad (25)\]

Define $p = b + \delta b$, $q = b - \delta b$. (25) becomes

\[-\frac{1}{3}K^2\bar{\theta}^3 - pkD^2 - k^2\frac{D^3}{3} < -p^2\bar{\theta}\]

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It is sufficient if we could prove:

\[ pkD^2 \geq p^2\tilde{\theta} \]

With definition of k, D and \( S_0 \)

\[ S_0(S_0 - \delta b) \geq p\tilde{\theta} \]

\[ (b + K\tilde{\theta})(b + K\tilde{\theta} - \delta b) \geq p\tilde{\theta} \tag{26} \]

For inequality (26), define \( G(\tilde{\theta}) \) as

\[ G(\tilde{\theta}) = \text{LHS} - \text{RHS} \]

To prove (26), it suffices to prove \( G(\tilde{\theta}) \geq 0 \). Since \( K = 1 + \frac{\delta b}{b+(1+\delta)\tilde{\theta}} > 1 \)

\[ G(\tilde{\theta}) > (\theta + b)(\tilde{\theta} + q) - p\tilde{\theta} = \tilde{\theta}^2 + b(1 - 2\delta)\tilde{\theta} + bq \]

However, if \( \delta \leq 0.5 \), then \( G(\tilde{\theta}) > 0 \), which concludes our proof. So we will focus on the case

\[ \delta > 0.5 \tag{27} \]

from now on.

\[ G(\tilde{\theta}) = K^2\tilde{\theta}^2 + [K(q + b) - p]\tilde{\theta} + qb \]

\[ G(0) = bq \geq 0 \]

To prove \( G(\tilde{\theta}) \geq 0 \), it is sufficient to prove

\[ G'(\tilde{\theta}) \geq 0 \]

\[ G'(\tilde{\theta}) = 2\tilde{\theta}K^2 + K(b + q) - p + 2\tilde{\theta}^2KK' + (b + q)\tilde{\theta}K' \]

By \( K(\tilde{\theta}) = 1 + \frac{\delta b}{b+(1+\delta)\tilde{\theta}} \), we have its derivative with regard to \( \tilde{\theta} \)

\[ K'(\tilde{\theta}) = -\delta b \frac{1 + \delta}{[b + (1 + \delta)\tilde{\theta}]^2} \]

\[ K''(\tilde{\theta}) = 2\delta b \frac{(1 + \delta)^2}{[b + (1 + \delta)\tilde{\theta}]^3} \]

Since \( G'(0) = K(b + q) - p = (1 + \delta)(b + q) - p = (1 + \delta)q \geq 0 \), to arrive at \( G'(\tilde{\theta}) \geq 0 \),
it is sufficient to prove
\[ G''(\bar{\theta}) \geq 0 \]  
\( G''(\bar{\theta}) = 4\tilde{\theta}KK' + 2K^2 + (b + q)K' + 2(2\tilde{\theta}KK' + \tilde{\theta}^2K'K' + \tilde{\theta}^2KK'') + (b + q)(K' + \tilde{\theta}K'') \)
\[ = 8\tilde{\theta}KK' + 2(b + q)K' + 2\tilde{\theta}^2K'K' + 2\tilde{\theta}^2KK'' + (b + q)\tilde{\theta}K'' \]

To prove (28), because \(2\tilde{\theta}^2K'K' + 2\tilde{\theta}^2KK'' > 0\) it is sufficient to prove
\[ 4\tilde{\theta}KK' + K^2 + \frac{b + q}{2}\tilde{\theta}K'' \geq 0 \]

Plug in \(K(\tilde{\theta})\), \(K'(\tilde{\theta})\) and \(K''(\tilde{\theta})\), it becomes
\[ (1 + \delta)^2(b + \tilde{\theta})^2 + \frac{b + q}{2}\tilde{\theta}^2b(1 + \delta)^2 \geq [4\tilde{\theta}K + (b + q)]\delta b(1 + \delta) \]
\[ (1 + \delta)(b + \tilde{\theta})^2 + \tilde{\theta}b(1 + \delta)(b + \tilde{\theta}) \geq [4\tilde{\theta}K + (b + q)]\delta b \]
\[ (1 + \delta)^2\tilde{\theta}^2 + \tilde{\theta}b[2(1 + \delta) + \frac{(b + q)\delta(1 + \delta)}{b + (1 + \delta)\tilde{\theta}} - 4K\delta] + (1 + \delta)b^2 - \delta b^2(2 - \delta) \geq 0 \]
\[ (1 + \delta)^2\tilde{\theta}^2 + \tilde{\theta}b[2(1 + \delta) + \delta(b + q - 4b - 4\tilde{\theta})(1 + \delta)] + (1 + \delta^2 - \delta)b^2 \geq 0 \]
\[ (1 + \delta)^2\tilde{\theta}^2 + \tilde{\theta}b(1 + \delta)[2 - \delta(2 + \delta)b + \frac{4\tilde{\theta}}{b + (1 + \delta)\tilde{\theta}}] + (1 + \delta^2 - \delta)b^2 \geq 0 \]  
\[ (29) \]

Analyze the term in the square bracket \(\frac{(2 + \delta)b + 4\tilde{\theta}}{b + (1 + \delta)\tilde{\theta}}\).

If \(\frac{2 + \delta}{1 + \delta} \leq \frac{4}{1 + \delta}\), then \(\delta \leq \delta^* \approx 0.56\)
\[ [2 - \delta(2 + \delta)b + \frac{4\tilde{\theta}}{b + (1 + \delta)\tilde{\theta}}] \geq 2 - \delta \frac{4}{1 + \delta} \geq 2 - \delta \frac{4}{1 + \frac{1}{2}} \geq 2 - \delta \frac{8\delta^*}{3} > 0 \]
where the second inequality is assured by (27). Therefore, this concludes our proof.

If \(\frac{2 + \delta}{1 + \delta} > \frac{4}{1 + \delta}\), \(\delta > \delta^* \approx 0.56\) and \[2 - \delta(2 + \delta)b + \frac{4\tilde{\theta}}{b + (1 + \delta)\tilde{\theta}} > 2 - \delta(2 + \delta)\]. For (29),
\[ LHS > (1 + \delta)^2\tilde{\theta}^2 + \tilde{\theta}b(1 + \delta)[2 - \delta(2 + \delta)] + (1 + \delta^2 - \delta)b^2 \]
Thus, it suffices to prove

\[(1 + \delta)\tilde{\theta}^2 + \tilde{\theta}b(1 + \delta)(2 - 2\delta - \delta^2) + (1 + \delta^2 - \delta)b^2 \geq 0 \tag{30}\]

If \(2 - 2\delta - \delta^2 \geq 0\), \(\delta \leq \delta^{**} \approx 0.73\), we would be done here. Otherwise, LHS of (30) reaches its minimum at \(\tilde{\theta} = \frac{b(\delta^2 + 2\delta - 2)}{2}\). (30) becomes

\[-\frac{b^2(1 + \delta)(\delta^2 + 2\delta - 2)^2}{4} + (1 + \delta^2 - \delta)b^2 \geq 0\]

\[4(1 + \delta^2 - \delta) - (1 + \delta)(\delta^2 + 2\delta - 2)^2 \geq 0\]

which holds not only for \(\delta > \delta^{**} \approx 0.73\), but for \(\forall \delta \in [0, 1]\).

![Figure 14: 4(1 + \delta^2 - \delta) - (1 + \delta)(\delta^2 + 2\delta - 2)^2](image)

**Step 4.** The above argument only works for \(\tilde{\theta} + D \leq 1\). Now we rule out the case for \(\tilde{\theta} + D > 1\). If so, agent with \(\theta = \tilde{\theta}\) would deviate to \(S = 0\) under off-equilibrium belief \(\theta = 1\) because

\[U^a(\tilde{\theta}) = -S^2(\tilde{\theta}) - \delta b^2 < -S_0^2 - \delta b^2 \leq -S_0^2 = -D^2 < -(1 - \tilde{\theta})^2\]

\[\leq -(1 - \tilde{\theta} - b)^2 \leq -\delta(1 - \tilde{\theta} - b)^2 = U^a_{\text{deviation}}\]

where the fourth inequality holds since \(\tilde{\theta} < \frac{1}{2} - b\) (by the Proof of Lemma 7). In addition, the agent will deviate to \(S = 0\) if off-equilibrium belief is \(\theta = 0\) (by the Proof of Lemma 7). Therefore, agent with \(\theta = \tilde{\theta}\) would deviate no matter what
off-equilibrium belief is, which means this is not a perfect Bayesian equilibrium.

B Online Appendix

B.1 Appended Proof of Theorem 2

Proof. Now we consider the case where \( a_1(\theta) \) is discontinuous. We endeavor to show that any discontinuous IC strategy is sub-optimal. Throughout this proof, it is very helpful to keep in mind the figures of integral curves, and we only consider \( S \) as a function of \( \theta \) to avoid any confusion. It suffices to consider the case \( k_1 > 0 \), since cases for \( k_1 > 0 \) and \( k_1 < 0 \) are symmetric around \( \theta \)-axis. If function \(|S(\theta)|\) is discontinuous, so does the sender’s payoff \( U_S(\theta) = -S(\theta)^2 - \delta(\theta - k_2\theta - b_2)^2 \), violating the indifference condition at the discontinuity. Thus, function \(|S(\theta)|\) must be continuous at all time. Then we can apply the DE (3) to \(|S(\theta)|\). In particular, if \( \{ \theta \in \mathcal{I} : S(\theta) \geq 0 \} \) is a dense set in \( \mathcal{I} \), we should have:

\[
|S(\theta)||\frac{d|S(\theta)|}{d\theta} + k_1| = \delta(k_2 - 1)\theta + \delta b_2; \quad (31)
\]

if \( \{ \theta \in \mathcal{I} : S(\theta) \leq 0 \} \) is a dense set in \( \mathcal{I} \), we should have:

\[
-|S(\theta)||\frac{-d|S(\theta)|}{d\theta} + k_1| = \delta(k_2 - 1)\theta + \delta b_2. \quad (32)
\]

Moreover, by Lemmas 3 and 4, discontinuity can only occur once and whenever it (let \( \tilde{\theta} \) denote the discontinuous point) occurs, \( S(\theta) \) jumps upwards\(^{17}\). And \( S(\theta) \) satisfies (32) for \( \theta < \tilde{\theta} \), (31) for \( \theta > \tilde{\theta} \). We shall continue to analyze case by case. Let \( S^*(\theta) \) denote the optimal continuous IC strategy identified in the proof of Theorem 2. Let \( \tilde{S}(\theta) \) denote any discontinuous IC strategy.

Case 1: \( r = 0 \)

(\( S^*(\theta) \) crosses \( \theta \)-axis at \((m, 0)\))

\[
k_1^2(\theta - m) + k_1S^* + \delta b_2 \ln(\delta b_2 - k_1S^*) = \delta b_2 \ln(\delta b_2) \quad (S^* \geq 0).
\]

When \( k_1 > 0 \) and \( b_2 > 0 \),\(^{18}\) SOC requires \( S \geq 0 \) or \( S \leq -k_1b_2 \). Although the SOC does not bind \( S^*(\theta) \), it binds the initial condition of any discontinuous IC strategy \( \tilde{S}(\theta) \) such that \( \tilde{S}(0) \leq -k_1b_2 \). Let \( S^-; \delta b_2 \ln(\delta b_2) \) denote the negative branch of

\(^{17}\)When \( k_1 < 0 \), it jumps downwards.\(^{18}\) The case for \( b_2 < 0 \) is centrosymmetric.
the integral curve where $S^*$ lies:

$$k_1^2(\theta - m) + k_1 S^- + \delta b_2 \ln(\delta b_2 - k_1 S^-) = \delta b_2 \ln(\delta b_2) \quad (S^- \leq 0).$$

For $\theta \in [m, \tilde{\theta}]$,

$$|\tilde{S}(\theta)| \geq |S^-(\theta)| \geq |S^*(\theta)|$$

where the second inequality is by the monotone expansion lemma. By $|\tilde{S}(\tilde{\theta})| \geq |S^*(\tilde{\theta})|$ and $|\tilde{S}(\theta)|$ being continuous,

$$|\tilde{S}(\theta)| \geq |S^*(\theta)|$$

for $\theta \in [\tilde{\theta}, M]$. Thus, $\tilde{S}$ is uniformly dominated by $S^*$.

**Case 2: $r > 0$**

If $\theta_0 \notin \Theta$, WLOG, we only consider $\theta_0 < m$. The argument is exactly the same as for case 1.

Suppose $\theta_0 \in \Theta$. If $w_1(m - \theta_0) < \tilde{S}(m) < w_2(m - \theta_0)$, we must have $\tilde{\theta} < \theta_0$, 

$$|\tilde{S}(\tilde{\theta})| < w_2(\tilde{\theta} - \theta_0)$$

by $w_1 > 0 > w_2$ and $|w_1| \leq |w_2|$. Thus, $\tilde{S}$ cannot be supported on $\Theta$. If $\tilde{S}(m) < w_1(m - \theta_0)$, $\tilde{S}$ either cannot be supported on $\Theta$ (if $|\tilde{S}(\tilde{\theta})| < w_2(\tilde{\theta} - \theta_0)$) or is uniformly dominated by $S^* = w_1(\theta - \theta_0)$ (if $|\tilde{S}(\tilde{\theta})| > w_2(\tilde{\theta} - \theta_0)$).

**Case 3: $r < 0$**
Figure 16: Case 2

Figure 17: Case 3.1
If \( \theta_0 \notin \Theta \), WLOG, we only consider \( \theta_0 > M \). The argument is exactly the same as for case 1.

Suppose \( \theta_0 \in \Theta \). By the monotone expansion lemma,

\[
|\tilde{S}(\tilde{\theta})| \geq |S^*(\tilde{\theta})|
\]

In case 3.1 or 3.2, if \(|\tilde{S}(\tilde{\theta})| \leq |w_2(\tilde{\theta} - \theta_0)|\),

\[
|\tilde{S}| > |S^*|
\]

\( \tilde{S} \) is uniformly dominated by \( S^* \); if \(|\tilde{S}(\tilde{\theta})| > |w_2(\tilde{\theta} - \theta_0)|\), \( \tilde{S} \) is either undefined for the entire \( \Theta \) or uniformly dominated by \( S^* \) by the monotone expansion lemma.

B.2 Analysis of Quadratic Linear Signaling Games with \( \Delta < 0 \)

Here we reestablish the result for quadratic linear signaling games with \( \Delta < 0 \). Throughout this section, we assume \( \Delta = k_1^2 + 4r < 0 \). Similar to the proof of Theorem 2, we only need to consider \( k_1 > 0 \). We divide the discussion into two cases.

Case 1: \( \theta_0 \notin \Theta \)

**Proposition 6.** Suppose \( \theta_0 \notin \Theta \). There exists a unique dominant separating strategy.
Moreover, it is non-linear, continuous, monotonic, and differentiable. Second, there exists a unique dominant PBE. Moreover, in this PBE, the sender takes the dominant strategy.

Proof of Proposition 6. Define $q \triangleq \frac{1}{2} \sqrt{-\Delta}$, $z \triangleq w + \frac{k_1}{2}$. We rewrite DE (12)

$$\frac{wdw}{(w + \frac{k_1}{2})^2 - \frac{\Delta}{4}} = -\frac{d\phi}{\phi}$$

$$\frac{zdz}{z^2 + q^2} - \frac{k_1}{2} \frac{dz}{z^2 + q^2} = -\frac{d\phi}{\phi}$$

The general solutions are

$$\ln |S^2 + k_1 S \phi - r \phi^2| - \frac{k_1}{q} \arg \tan(\frac{S}{q \phi} + \frac{k_1}{2q}) = C \quad (33)$$

Let $g(\phi, S) \triangleq \ln |S^2 + k_1 S \phi - r \phi^2| - \frac{k_1}{q} \arg \tan(\frac{S}{q \phi} + \frac{k_1}{2q})$. If $\theta_0 > M$, the unique dominant solution crosses $(\theta, S) = (m, 0)$:

$$g(\phi, S) = \ln(-r(m - \theta_0)^2) - \frac{k_1}{q} \arg \tan(\frac{k_1}{2q}) \quad (S \geq 0, \phi \in [m - \theta_0, M - \theta_0]) \quad (34)$$

which satisfies the SOC, is continuous and non-linear. It is monotonic by $\frac{d\theta_0}{d\theta} = \frac{r\phi}{S} > 0$.

If $\theta_0 < m$, the unique dominant one crosses $(\theta, S) = (M, 0)$:

$$g(\phi, S) = \ln(-r(M - \theta_0)^2) - \frac{k_1}{q} \arg \tan(\frac{k_1}{2q}) \quad (S \leq 0, \phi \in [m - \theta_0, M - \theta_0]) \quad (35)$$
which satisfies the SOC, is continuous and non-linear. It is monotonic by \( \frac{da_1}{d\theta} = \frac{r\phi}{S} > 0 \).

Case 2: \( \theta_0 \in \Theta \)

**Proposition 7.** Suppose \( \theta_0 \in \Theta \). There is no continuous IC strategy.

*Proof of Proposition 7.* All solutions of (33) features \( S(\theta_0) \neq 0 \). By \( \frac{da_1}{d\phi} = \frac{r\Phi}{S} \), any strategy continuous at \( \theta_0 \) with \( S(\theta_0) \neq 0 \) must violate monotonicity.

Thus, an IC strategy must jump at \( \theta_0 \). By proposition 1, it is discontinuous only at \( \theta_0 \) and jumps upward.

**Proposition 8.** Suppose \( \theta_0 \in \Theta \). IC separating strategies exist if and only if \( k_2 > 0 \).

*Proof of Proposition 8.* If \( k_2 \leq 0 \), the SOC implies

\[
\frac{\Phi}{S} \leq 0.
\]

No separating strategy can jump upward. Thus, no IC separating strategies exists.

If \( k_2 > 0 \), SOC is

\[
\frac{S}{\Phi} \geq \frac{-k_1(k_2 - 1)}{k_2}
\]

by \( k_2 < 1 \) due to \( \Delta < 0 \). Then there exists discontinuous solutions of (33) jumping upward at \( \theta_0 \) while satisfying the SOC. To show IC, it suffices to show any sender with \( \theta \in [m, \theta_0] \) does not mimic \( \theta' \in (\theta_0, M] \). By \( k_1 > 0 \), \( k_2 > 0 \), \( |S(\theta_0^-)| = |S(\theta_0^+)| \). \( \theta \) mimicking \( \theta' \) is worse than mimicking \( 2\theta_0 - \theta' \) for both periods. IC separating strategies thus exist.

Among all IC separating strategies, the dominant one is closet to \( S = 0 \) and satisfies

\[
S(\theta = m) = \frac{-k_1(k_2 - 1)}{k_2}(m - \theta_0)
\]

if \( \theta_0 \geq \frac{m + M}{2} \), satisfies

\[
S(\theta = M) = \frac{-k_1(k_2 - 1)}{k_2}(M - \theta_0)
\]

if \( \theta_0 < \frac{m + M}{2} \).

From now on, we focus on \( k_2 > 0 \). \( \Delta = k_1^2 + 4\delta(k_2 - 1) < 0 \) implies \( k_2 < 1 - \frac{k_1^2}{4\delta} \). Thus, \( k_2 > 0 \) further implies \( k_2 \in (0, 1 - \frac{k_1^2}{4\delta}) \) which is a very small parameter space. We next show in most cases, no PBE exists.
Proposition 9. Suppose $\theta_0 \in \Theta$ and $k_2 > 0$. If $k_2 < \frac{k_1}{\sqrt{\delta}}$, there is no PBE.

Proof of Proposition 9. WLOG, suppose $\theta_0 \geq \frac{m+M}{2}$, SOC requires

$$S(\theta = m) \leq \frac{-k_1(k_2 - 1)}{k_2} (m - \theta_0)$$

$$a_1(\theta = m) \leq k_1 m + b_1 - \frac{k_1(k_2 - 1)}{k_2} (m - \theta_0)$$

By $\Delta < 0, r < 0, \frac{\phi}{S} > 0, \frac{d a_1}{d \phi} = \frac{r \phi}{S}$, $a_1$ is decreasing whenever continuous. Thus,

$$a_1(\theta = \theta_0) < a_1(\theta = m) \leq k_1 m + b_1 - \frac{k_1(k_2 - 1)}{k_2} (m - \theta_0),$$

$$|S(\theta = \theta_0)| > k_1(\theta_0 - m) + \frac{k_1(k_2 - 1)}{k_2} (m - \theta_0) = \frac{k_1}{k_2} (\theta_0 - m).$$

If the sender with $\theta = \theta_0$ deviates to $a_1' = k_1 \theta_0 + b_1$, the first-period gain is at least $[\frac{k_1}{k_2} (\theta - m)]^2$. Yet, the largest second-period punishment is $\delta (\theta_0 - m)^2$. Therefore, if $(\frac{k_1}{k_2})^2 > \delta$, the sender with $\theta = \theta_0$ deviates for sure and there is no PBE. \qed